The Sturm Liouville problem and diffusion modeling

Il problema di Sturm Liouville e la costruzione di modelli per processi di diffusione

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Riassunto: I lavoro analizza la connessione tra il problema agli autovalori che caratterizza i generatori di una classe di processi di Markov e un noto risultato circa le NEF con funzione di volatilita' quadratica.

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1. Sturm Liouville problem, Markov generators and NEF

We begin with three well known facts.

First. The classical Sturm-Liouville (SL) problem concerns solutions of a second order linear differential equation: $a(x)\frac{\partial^2}{\partial x^2}u(x)+b(x)\frac{\partial}{\partial x}u(x)-\lambda u(x) = 0$ in a a particular self adjoint form. A classical problem is that of finding all possible polynomial solutions of such equation. It is clear that u can be a polynomial in x only if a and b are polynomials where the maximum degree for a is 2 and for b is 1. In (Bochner 1929) a theorem stated that essentially (i.e. modulus a set of transforms) only 3 possible solutions existed such that the polynomial system generated by the solutions was infinite (that is polynomials of any degree are solutions). These polynomial solutions are known as the Jacobi, Hermite and Laguerre polynomials. The weight functions of these polynomials are the Beta, the Gaussian and the Gamma families. In an analogous way it is possible to prove that, when the first and second derivatives are approximated with first (Δ) and second ($\Delta\nabla$) differences and the argument of the solutions are restricted to integers, again essentially only three polynomial solutions with integer argument of the problem $a(x)\Delta\nabla u(x) + b(x)\Delta u(x) - \lambda u(x) = 0$ exist: Meixner, Krawtchouk and Charlier. The corresponding weight functions are the Negative Binomial, the Binomial and the Poisson.

Second. It has been known at least since the fifties that only three generators of time homogeneous Markov diffusions and correspondingly only three generators of birth and death Markov processes allow for a complete system of polynomials as eigenfunctions, these polynomials are, resp., the Jacobi, Hermite, Laguerre, Meixner, Krawtchouk and Charlier. The invariant distributions of these processes are, resp., the Beta, the Gaussian, the Gamma, the Negative Binomial, the Binomial and the Poisson.

Third. In (Morris 1982) is proved that only (essentially) six elements in the Natural Exponential Family (NEF) are such that the variance is an at most quadratic function of the mean. These are the Gamma, the Normal, the Poisson, the Binomial, the Negative Binomial and the distribution of the natural observable of the Beta.

The first and second of these results are easy to connect, as the eigenproblem for the generators of the above mentioned processes is equivalent to the SL problem described above (e.g. in the case of diffusions $a(x) = \sigma(x)^2/2$ one half the square of the diffusive term and $b(x) = \mu(x)$ the drift term).

While Morris (1982) contains a short discussion of the polynomials generated by the six quadratic NEF, as far as I know a direct connection between the SL problem and the Morris results concerning quadratic NEF has still to be discussed. The result is interesting, for instance, as it allows for the application of a number of results connected with limit behavior of Markov processes to the study of NEF as, for instance, the study of martingales generated by the natural sufficient statistics of the exponential family under simple random sampling.

2. An equivalence result

In this section we sketch (due to lack of space) the proof of the following result which, in practice, shows how (Morris 1982) results are implied in (Bochner 1929).

Theorem *The polynomial SL and the (Morris 1982) problem ere equivalent. The weight function of each SL solution is an element of the exponential family and the six solutions in (Morris 1982) are the distributions of the natural observables of these.*

The proof is similar for the continuous and the discrete case, we sketch the former.

The first step is to show that $w(x)$ is the density of the weight function for the SL polynomials iff it has the shape: $w(x)a(x) = c_0 e^{-\int^x b(z)/a(z)dz}$ with a an at most second degree and b an at most first degree polynomial. This result derives at once from the fact that w solves $(a(x)w)' + b(x)w = 0$. The second step is to show that, if we consider equivalent densities that can be derived one from the other through change of origin, scale and convolution only essentially three cases of this density exist. This is the lengthy part of the proof as all possible cases must be taken into account and discussed. This step is analogous to the discussion in (Morris 1982) about the possible values of the coefficient in an at most quadratic variance function. The third step is to show that each of the allowable solutions $w(x)a(x) = c_0 e^{-\int^x b(z)/a(z)dz}$ is a member of the exponential family and to derive the distribution of the corresponding natural observable. It is at this point that the Beta case is transformed in the ”sixth family” (the generalized Hyperbolic Secant), in fact the Beta is the only case where the natural observable is not a linear function of x. The last step, quite direct, is to show that the variance functions of the distributions of each of these observables is quadratic (actually, this does not even require the computation of the density of the distribution of the natural observables).

References

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