An income mobility measure based on Zenga's inequality index

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Abstract This paper proposes a new measure of income mobility in the form of re-ranking which occurs in the move from an initial income distribution to a final income distribution. The re-ranking measure is based on Zenga's new index and concentration curve. The measure summarizing the overall re-ranking is expressed as the average of various point re-ranking measures calculated across the income distribution.

1 Introduction

Reshuffling of individuals in the move from one income distribution to another is a phenomenon of interest for literature on income taxation, economic growth, equality of opportunity. A widely-used income mobility measure is the Atkinson-Plotnick re-ranking measure (Plotnick, 1981) (hereafter, R^{AP}), which is based on the Gini index and the underlying Lorenz curve. However, statistical literature offers various alternative inequality indices for which, to our knowledge, analogous type of re-ranking measure has not been developed. Among the inequality indices competing with the Gini index there is a recent proposal by Zenga (2007), which has drawn increasing interest due to its straightforward interpretability and desirable properties (Greselin *et al*., 2010). This paper suggests a new measure of income mobility in the form of re-ranking which is based on Zenga's new approach to inequality measurement. Rank-changes of individuals are captured by comparing the mean incomes of two disjoint and exhaustive parts of population, the composition of which varies across the quintiles of the income distribution. Here we focus on the methodological side of the proposed re-ranking measure, whereas future work will be devoted to illustrate the areas of application and the empirical outcomes resulting from the application to real income data.

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2 Definition and Notation

Consider a population of n individuals and let $\mathbf{y} = (y_1, \dots, y_n)'$ be the income vector where incomes are arranged in increasing order. For the aforementioned discrete distribution the cumulative population share is $p = i/n$ and it denotes the population share of individuals whose incomes are less than or equal to y_i. To introduce the Zenga inequality index (Zenga, 2007; Zenga *et al*., 2011), we define the lower mean at $p = i/n$ as the arithmetic mean of incomes less than or equal to y_i

$$\overset{-}{M}_i = \frac{1}{i}\sum_{j=1}^{i} y_j \ , \tag{1}$$

and the upper mean as the arithmetic mean of the remaining part of the income distribution

$$\overset{+}{M}_i = \begin{cases} \dfrac{1}{n-i}\sum_{j=i+1}^{n} y_j & for \quad i = 1, \dots, n-1 \\ y_n & for \qquad i = n \end{cases} . \tag{2}$$

Then, the Zenga inequality measure for $p = i/n$ is given by

$$I_i = \frac{\overset{+}{M}_i - \overset{-}{M}_i}{\overset{+}{M}_i} \ , \tag{3}$$

The point inequality measure in (3) ranges from 0 (when the lower mean equals the upper mean) to 1 (when the lower mean equals 0). The index summarizing the inequality across the income distribution is

$$I = \frac{1}{n}\sum_{i=1}^{n} I_i \ . \tag{4}$$

I is bounded below by 0 (all individuals receive the same income) and above by $1 - 1/n^2$ (one individual receives total income).

3 A New Re-ranking Measure

Consider now an initial period (hereafter, period 0) and a final period (henceforth, period 1) and suppose that individuals receive income in both the periods. Let $\mathbf{y}_1 = (y_{1,1}, \dots, y_{1,n})'$ stand for the income vector with period 1 incomes sorted in increasing order, and I^1 stand for the period 1 Zenga index. We define the vector $\mathbf{y}_{1|0} = (y_{1|0,1}, \dots, y_{1|0,n})'$ which includes the period 1 incomes lined up by ascending order of their corresponding period 0 incomes. By replacing y_j in (1) with the *j*-th element of $\mathbf{y}_{1|0}$, one obtains the period 1 mean income of the poorest 100(i/n)% part of population in period 0,

$$\overset{-}{M}_{1|0,i} = \frac{1}{i}\sum_{j=1}^{i} y_{1|0,j} \ . \tag{5}$$

The period 1 mean income of the richest 100(1- i/n)% of population in period 0 is

$$\overset{+}{M}_{1|0,i} = \frac{1}{n-i}\sum_{j=i+1}^{n} y_{1|0,j} \;. \qquad (6)$$

The concentration measure at $p = i/n$ is given by

$$I_i^{1|0} = \frac{\overset{+}{M}_{1|0,i} - \overset{-}{M}_{1|0,i}}{\max\left(\overset{+}{M}_{1|0,i}, \overset{-}{M}_{1|0,i}\right)} \quad for \quad i = 1,\dots,n-1 \,. \qquad (7)$$

Different from the inequality measure in (3), the concentration measure in (7) is calculated for i ranging from 1 to $n-1$; since the goal here is to capture rank-changes between individuals, in the following we show that taking into account $n-1$ comparisons between I_i^1 and $I_i^{1|0}$ (with $i = 1,\dots,n-1$) is enough to reveal all the possible re-ranking cases. Since period 1 incomes are not necessarily arranged in increasing order in $\mathbf{y}_{1|0}$, $\overset{-}{M}_{1|0,i}$ may be greater than $\overset{+}{M}_{1|0,i}$. In (7), dividing the difference $\overset{+}{M}_{1|0,i} - \overset{-}{M}_{1|0,i}$ by the maximum between $\overset{+}{M}_{1|0,i}$ and $\overset{-}{M}_{1|0,i}$ ensures that $I_i^{1|0}$ ranges from $-I_{n-i}^1$ to I_i^1; being I_{n-i}^1 and I_i^1 the period 1 inequality measures for cumulative population shares $(1-i/n)$ and i/n, respectively. $I_i^{1|0}$ equals:

- $-I_{n-i}^1$ when the poorest i individuals in period 0 income parade (their reciprocal positions do not matter) receive the highest i incomes in period 1 income parade; that is, when the incomes filling the first i positions in $\mathbf{y}_{1|0}$ belong to the same individuals whose incomes are arranged in the last i positions in $\mathbf{y}_1$. Thus, $\overset{-}{M}_{1|0,i} = \overset{+}{M}_{1,n-i}$ and $\overset{+}{M}_{1|0,i} = \overset{-}{M}_{1,n-i}$; these equalities imply that $I_i^{1|0} = -I_{n-i}^1$. If the ranking of individuals in the period 1 income parade is exactly the opposite compared to the one in period 0 income parade, it immediately follows that $I_i^{1|0} = -I_{n-i}^1$ for all i (with $i = 1,\dots,n-1$).
- I_i^1 when the poorest i individuals in period 0 income parade (regardless of their reciprocal positions) earn the lowest i incomes in period 1 income parade. Indeed, it straightforward to check that $I_i^{1|0} = I_i^1$. If every individual maintains unaltered his rank when moving from period 0 to 1, one has $I_i^{1|0} = I_i^1$ for all i (with $i = 1,\dots,n-1$) since $\mathbf{y}_{1|0}$ coincides with $\mathbf{y}_1$.

The point re-ranking measure is given by

$$R_i = I_i^1 - I_i^{1|0}, \qquad (8)$$

and it varies in the interval $\left[0, I_i^1 + I_{n-i}^1\right]$. R_i can be interpreted as follows:

- If $R_i = 0$, no re-ranking occurs between members of the poorest 100(i/n)% of population in period 0 and those of the richest 100(1-i/n)% of population in the same period (i.e., $I_i^{1|0} = I_i^1$).

- If $0 < R_i < I_i^1$, re-ranking occurs between individuals of the poorest 100(*i*/*n*)% of population in period 0 and those of the richest 100(1-*i*/*n*)% of population in the same period (i.e., $0 < I_i^{1|0} < I_i^1$).
- If $I_i^1 < R_i < I_i^1 + I_{n-i}^1$, re-ranking exists between individuals of the poorest 100(*i*/*n*)% of population in period 0 and those of the richest 100(1-*i*/*n*)% of population in period 0. Moreover, re-ranking of the two subgroup mean incomes occurs when passing from period 0 to 1 (i.e., $-I_{n-i}^1 < I_i^{1|0} < 0$).

By analogy with the index I in (4), the synthetic measure of re-ranking is given by

$$R = \frac{1}{n-1}\sum_{i=1}^{n-1} R_i \ . \tag{9}$$

R summarizes the reshuffling of individuals in the distribution. Similar to R^{AP}, R is nonnegative and it is bounded above by two times the relative inequality index ($2\tilde{I}^1$, as R^{AP} maximum is two times the period 1 Gini index)[1] and below by zero. R differs from R^{AP} since the former can be seen as the average of $n-1$ between-group re-ranking measures. Indeed, R_i is sensitive to re-ranking between any two members of different parts of population (one belonging to the poorest 100(*i*/*n*)% of population in period 0, the other belonging to the richest 100(1-*i*/*n*)% of population in period 0), but it ignores re-ranking between members of the same part of population.[2] The $n-1$ between-group re-ranking measures can be plotted against cumulative population shares, depicting a re-ranking curve; thus, one can detect the intervals of cumulative population share where re-ranking lies below or above the overall re-ranking in (9).

References

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[1] If the period 1 ranking of individuals is the opposite of the period 0 ranking of individuals, then $R_i = I_i^1 + I_{n-i}^1$ for all i. Hence, $R = \frac{1}{n-1}\sum_{i=1}^{n-1} I_i^1 + \frac{1}{n-1}\sum_{i=1}^{n-1} I_{n-i}^1 = 2\tilde{I}^1$, where $\tilde{I}^1$ slightly differs from I^1 since the former is in the interval $[0,1]$, satisfying the principle of normalization but not the principle of population replication (see Zenga, 2007).

[2] For instance, suppose the individual A has rank *j* and the individual B has rank *j*-1 in the period 0 income parade; then, assume they exchange their positions in period 1 income parade whereas the positions of the remaining individuals are unaltered. It is straightforward to verify that $\overset{-}{M}_{1|0,i}$ ($\overset{+}{M}_{1|0,i}$) equals $\overset{-}{M}_{1,i}$ ($\overset{+}{M}_{1,i}$) at any i for which the two aforementioned individuals are included in the same part of population in period 0; that is, the only nonzero re-ranking measure is R_{j-1}.