On the stationarity of the Threshold Autoregressive process: the two regimes case

Francesco Giordano, Marcella Niglio and Cosimo Damiano Vitale

Abstract This paper deals with the stationarity of the nonlinear Threshold Autoregressive process (TAR) whose self exciting representation (shortly called SETAR) has been widely presented in [8] and [9]. Starting from these results, the main aim is the discussion of some theoretical differences between TAR and SETAR models, mainly related to their stationarity. We shortly provide new issues on the stationarity of the TAR model whereas those results are discussed and compared through empirical examples with what well known in the SETAR context.

Key words: Threshold autoregressive process, stationarity

1 The Threshold AutoRegressive processes (TAR)

Let $\{X_t\}$ a real time series with $t \in \mathscr{N}$, it is said to follow a TAR(ℓ, p) model if:

$$X_t = \sum_{k=1}^{\ell} \left[\phi_1^{(k)} X_{t-1} + \ldots + \phi_p^{(k)} X_{t-p} + a_t \right] I(Y_{t-d} \in \mathscr{R}_k), \tag{1}$$

where d is the threshold delay, Y_{t-d} is the threshold variable, $\mathscr{R}_k = (r_{k-1}, r_k]$ is a subset of the real line $\mathscr{R}$ such that $\bigcup_{k=1}^{\ell} \mathscr{R}_k = \mathscr{R}$ with $-\infty = r_0 < r_1 < \ldots < r_{\ell-1} <$

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$r_\ell = \infty$ and $I(\mathscr{A})$ is an indicator function which assumes value 1 on the subset $\mathscr{A}$. Further, in model (1), $\{a_t\}$ is assumed to be a sequence of independent and identically distributed random variables, with zero mean and $E[a_t^2] = \sigma^2 < \infty$.
It can be noted that if the following condition holds:

A1) the threshold variable is independent of X_t and a_t

then model (1) can be ascribed to the class of Random Coefficients Autoregressive models widely presented in [6] whose theoretical results cannot be directly applied to model (1) because the assumption on the independence of the coefficients given in [6] is not fulfilled by the TAR structure.
To better clarify this statement, consider an alternative representation of model (1) with $k = 2$. Let $\phi_i^{(1)} = \phi_i + \psi_i^{(1)}$, for $i = 1, 2, \ldots, p$, then model (1) becomes:

$$X_t = \sum_{i=1}^{p} \left(\phi_i + \psi_i^{(1)} I(Y_{t-d} \in \mathscr{R}_1) \right) X_{t-i} + a_t, \tag{2}$$

whose vector representation is:

$$\mathbf{X}_t = (\boldsymbol{\Phi} + \boldsymbol{\Psi}^{(1)} I(Y_{t-d} \in \mathscr{R}_1))\mathbf{X}_{t-1} + \mathbf{a}_t, \tag{3}$$

where $\mathbf{X}_t = (X_t, X_{t-1}, \ldots, X_{t-p+1})'$, $\mathbf{a}_t = (a_t, \mathbf{0}_{[1\times(p-1)]})'$ whereas

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ & \underset{[(p-1)\times(p-1)]}{\mathbf{I}} & & \underset{[(p-1)\times 1]}{\mathbf{0}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Psi}^{(1)} = \begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_p^{(1)} \\ & \underset{[(p-1)\times(p-1)]}{\mathbf{I}} & & \underset{[(p-1)\times 1]}{\mathbf{0}} \end{bmatrix},$$

with $\mathbf{I}$ and $\mathbf{0}$ the identity matrix and a null vector respectively.

Is is well known that different features of a stochastic process can be investigated through its moments. If we consider model (3) the expectation

$$E[\mathbf{X}_t] = (\boldsymbol{\Phi} + \boldsymbol{\Psi}^{(1)}\lambda)E[\mathbf{X}_{t-1}] \qquad (\text{with } \lambda = E[I(Y_{t-d} \in \mathscr{R}_1)])$$

is finite if proper conditions are provided on the matrices of coefficients of both regimes. More precisely it can be shown that if the two regimes have matrix of coefficients with dominant eigenvalues less than 1 then process (3) does not explode in mean. This statement has heavy consequences on the stationarity of model (3) and further remarks the difference of the TAR model with respect the Self Exciting Autoregressive (SETAR) process.

In fact, these last results need to be revised if we consider the SETAR model that is characterized by a threshold structure governed by the process X_t itself with delay d (in other words, $Y_{t-d} = X_{t-d}$). Note that in this case the first part of assumption (A1) cannot be longer true and even the results of [6] cannot be used.
The stationarity of this class of models has been differently investigated: the seminal contributions on the strict stationarity and ergodicity of the SETAR model are given in [7], [2], [3]. Their results are mainly focused on SETAR models with autoregressive regimes of order $p = 1$ whereas [1] and [5] then generalize those results in a wider context with $p \geq 1$ prividing the following stationarity conditions:

$$\max_k \sum_{i=1}^{p} |\phi_i^{(k)}| < 1 \qquad \text{(as shown in [1])} \tag{4}$$

$$\sum_{i=1}^{p} \max_k |\phi_i^{(k)}| < 1 \qquad \text{(as shown in [5])}. \tag{5}$$

In the following section, starting from the presentation of some case study, we discuss these conditions giving empirical evidence of the difference between the stationarity of the TAR and SETAR models.

2 Discussion on the stationarity of the TAR and SETAR models

In order to give an idea of how different could be the stationarity conditions between the TAR and the SETAR model consider the following two examples.
Example 1. Let $X_{1,t} \sim TAR(2;2)$ with $\phi_1^{(1)} = 0.6$, $\phi_2^{(1)} = -0.9$, $\phi_1^{(2)} = -0.9$, $\phi_2^{(2)} = -0.2$, $d = 1$, threshold value $r_1 = 0$ and threshold variabile $Y_t \sim AR(1)$ with autoregressive parameter $\phi_1 = 0.7$, and let $X_{2,t} \sim SETAR(2;2)$ with the same autoregressive parameters, $d = 1$ and $r_1 = 0$. In Figure 1 are shown the traces of the simulated TAR (left) and SETAR (right) processes. It is evident the completely different behavior of the two time series and how the local stationarity of the AR regimes does not guarantee the global stationarity of the SETAR process. Completely different results can be observed for the TAR case.

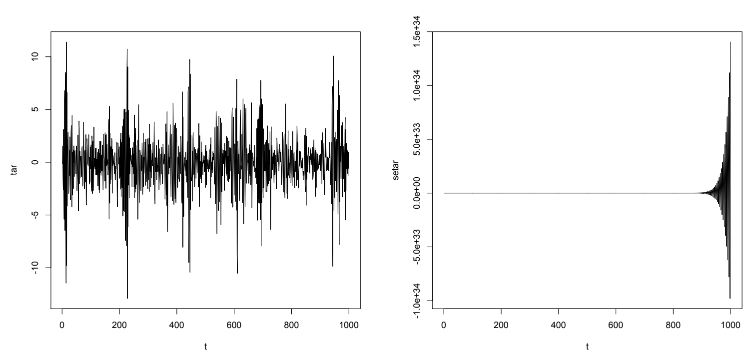

Fig. 1 Traces of the simulated TAR (left) and SETAR(right) processes of Example 1

Example 2. In this second case study the autoregressive parameters of both models are with $\phi_1^{(1)} = -1.6$, $\phi_2^{(1)} = -1.1$, $\phi_1^{(2)} = -2.1$, $\phi_2^{(2)} = -0.8$ and the threshold variable follows the same process of Example 1. The traces of the TAR and SETAR model are now presented in Figure 2 (left and right respectively) where it can be noted that in presence of two nonstationary regimes the SETAR model does not explode whereas the TAR process has a clear explosive behavior.

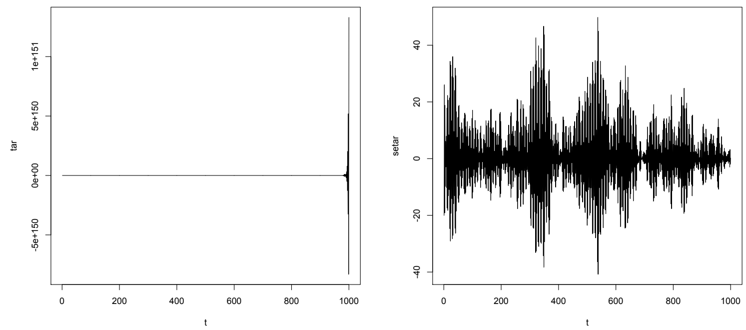

Fig. 2 Traces of the simulated TAR (left) and SETAR(right) processes of Example 2

These two examples show how the stochastic structure of the two models under analysis is different and how the feedback of the threshold variable in the SETAR case impacts the statistical properties of model (3). More precisely it can be shown that when the two regimes are locally stationary the TAR process cannot explode whereas when two regimes are not stationary, the behavior of the TAR process is related to the generating process of the threshold variable.

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