Social welfare orderings of the Generalized-Lorenz Type: applications of an extended equivalence theorem

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Abstract An appropriate way of ranking distributions from a social welfare viewpoint is the Generalized Lorenz ordering GL. A large family of social welfare functions have the form of expected utilities. This is possible because under general conditions these indicators are consistent with GL, namely there is a ”bridge” between the theory of Expected Utility and decision making under risk and the theory of social choice. The purpose of this presentation is to point out that this bridge can be widened by means of an extended equivalence Theorem that allows one to import a wide class of social indicators from utilities (and viceversa) under more general orderings that account for satisfaction, deprivation etc. This is related to Yaari's dual theory of rank-dependent social welfare.

Key words: Stochastic orderings, Duality, Social choice

1 Generalized Lorenz for ordering social welfare

A social welfare indicator W measures the distribution in a population of some quantity X related to individual well-being - wealth, income, consumption, health, education, quality of life or a combination thereof. Going back to the seminal paper by A.B. Atkinson (1970, there is a general consensus that any W should be a function of the cumulative frequency distribution (cdf) $F(x)$ of X, which increases if X increases for at least one individual of the population, and/or if X becomes more equally distributed by means of a sequence of Pigou-Dalton transfers. Thus an appropriate way of ranking distributions from a social welfare viewpoint seems to be the *Generalized Lorenz ordering* $\leq^{GL}$:

Definition 1. $X_1 \leq^{GL} X_2$ (X_2 is GL-better than X_1) if and only if

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$$\int_0^p F_1^{-1}(\alpha)d\alpha \leq \int_0^p F_2^{-1}(\alpha)d\alpha \qquad \forall\, p\ \in [0,1], \tag{1}$$

where $X \sim F_1, Y \sim F_2$, and F^{-1} denotes the *quantile function* of a cdf F, i.e.

$$F^{-1}(\alpha) = \inf\{x : F(x)_{\geq}\alpha\}, \qquad \alpha \in [0,1]; \tag{2}$$

Roughly speaking, $X \leq^{GL} Y$ means that Y can be obtained from X first by giving rise to a variable V greater than X with respect to first order stochastic dominance and with $E(V) = E(Y)$, and then making the distribution of V more equal, so that $V \leq^{L} Y$, with $\leq^{L}$ the usual *Lorenz ordering*. In the case of finite populations of the same size n, $\leq^{GL}$ is the reverse of *supermajorization* of vectors: for a modern presentation see Marshall, Olkin and Arnold (2011).

A large family of social welfare functions are the so-called *utilitarian* ones, which, (up to an increasing[1] transform), are of the form $E\left(u(X)\right)$ for a given utility function $u(\cdot)$, whose concavity reflects *inequality aversion*. This is possible because under general conditions indicators of the expected utility form are consistent with $\leq^{GL}$ (see Shaked and Shanthikumar, 2007):

Theorem 1.

$$\int_0^p F_1^{-1}(\alpha)d\alpha \leq \int_0^p F_2^{-1}(\alpha)d\alpha \qquad \forall\, p \in [0,1)$$
$$\Longleftrightarrow\ E\left(u(X)\right) \leq E\left(u(Y)\right) \qquad \forall\, u(\cdot) \text{ concave and increasing.}$$

Theorem 1 acts as a "bridge" between the theory of Expected Utility and decision making under risk and the theory of social choice. The purpose of this presentation is to point out that this bridge can be widened. Theorem 2 in Section 2 extends the equivalence of Theorem 1, and thus allows one to import a wider class of social indicators from utilities (and viceversa). Furthermore, there is also a dual theory of rank-dependent social welfare, due to Yaari (1987), with indicators of the form

$$Y(F) = \int_{-\infty}^{\infty} x df(F(x)) = \int_0^1 F^{-1}(\alpha)df(\alpha) \tag{3}$$

where $f : [0,1] \to [0,1]$ is an increasing function with $f(0) = 0$ and $f(1) = 1$. The "strength" of convexity of f is a degree of inequality aversion. Special cases are the *S-Gini functions*, where $f(p) = p^{\rho}$ with $\rho > 1$. When $f(p)$ is differentiable, the intuitive meaning of $Y(F)$ is that a positive weight $w(p) = \frac{\partial f}{\partial p}$ is attached to all income ranks p. It is possible to check that $Y(F)$ is consistent with the $\leq^{GL}$-ordering. In Section 3 we shall show some consequences of Theorem 2 for this type of indices as well.

[1] Throughout the paper increasing means non-decreasing, and similarly for decreasing

2 The extended equivalence theorem

Our stochastic orderings involve two types of "distortions". One takes the form of a direct transformation of the cdf: $v(F)$, for some function $v(\cdot)$. In the literature, cdf distortions have been described as "perceptions" and "probability weighting" . The other type of distortion is applied to the base measure, so that $\int g(x)dx$ becomes the Lebesgue-Stieltjes integral $\int g(x)du(x)$, sometimes shortened to $\int g(x)du$. Similarly for F^{-1}. For mathematical simplicity, all distributions under consideration will be assumed to have bounded support, although the theory applies more in general. More mathematical details are given in Giovagnoli and Wynn (2011).

We start with two base (distortion) functions u_0 and v_0

(i) $u_0 : \mathbf{R} \to \mathbf{R}$ is increasing and left continuous,

(ii) $v_0 : [0,1] \to [0,1]$ is increasing, right continuous, and $v_0(0) = 0, v(1^-) = 1$.

We recall that given a function $u_0 : \mathbf{R} \to \mathbf{R}$, a function $u(x)$ is called u_0-concave if $u(x) = \int_{-\infty}^{x} k(s)du_0(s)$ for some bounded decreasing function $k(x)$ on $\mathbf{R}$. Also, u is u_0-concave if and only if $u(x) = \phi(u_0(x))$ for some concave increasing function ϕ. Similarly for v_0-concave functions on $[0,1]$. By U^0 and V^0 we denote the classes of increasing u_0-concave and v_0-concave functions respectively.

Theorem 2. *Let F_1, F_2 be two cdf's on $\mathbf{R}$ satisfying the conditions*

$$(a) \int_{-\infty}^{\infty} |u_0(x)| dv_0(F(x)) < \infty$$
$$(b) \int v_0(\alpha) du_0(F^{-1}(\alpha)) < \infty$$

Then all the following statements are equivalent:

$$\int_{-\infty}^{c} v_0(F_1(x))du_0 \geq \int_{-\infty}^{c} v_0(F_2(x))du_0 \qquad \forall\, c \in \mathbf{R}$$
$$\int_{0}^{p} u_0(F_1^{-1}(\alpha))dv_0 \leq \int_{0}^{p} u_0(F_2^{-1}(\alpha))dv_0 \qquad \forall\, p \in [0,1)$$
$$\int_{-\infty}^{\infty} v_0(F_1(x))du(x) \geq \int_{-\infty}^{\infty} v_0(F_2(x))du(x) \qquad \forall\, u \in U^0$$
$$\int_{0}^{1} u_0(F_1^{-1}(\alpha))dv(\alpha) \leq \int_{0}^{1} u_0(F_2^{-1}(\alpha))dv(\alpha) \qquad \forall\, v \in V^0$$
$$\int_{-\infty}^{\infty} u(x)dv_0(F_1(x)) \leq \int_{-\infty}^{\infty} u(x)dv_0(F_2(x)) \qquad \forall\, u \in U^0$$
$$\int_{0}^{1} v(\alpha)du_0(F_1^{-1}(\alpha)) \geq \int_{0}^{1} v(\alpha)du_0(F_2^{-1}(\alpha)) \qquad \forall\, v \in V^0.$$

Theorem 1 is clearly a special case of Theorem 2 when both u_0 and v_0 are the identity.

3 Some applications to social welfare measures

Functionals of the form

$$\int_{-\infty}^{\infty} u(x)df(F(x)) \tag{4}$$

where u is a utility function and $f:[0,1]\to[0,1]$ is increasing and continuous, are the building blocks of the *rank-dependent expected utility* (RDEU) theory developed by Quiggin (1983) and later incorporated into Cumulative Prospect Theory by Tversky and Kahneman (1992). A simple consequence of Theorem 2 is obtained when u_0 is the identity and f_0 any given increasing function on $[0,1]$ standing for a "perception" :

F_2 is preferred to F_1 by all the Yaari welfare measures (3) for which the function f indicates a degree of inequality aversion at least as great as f_0 if and only if F_2 is preferred to F_1 by all utilitarian welfare indicators of the form $\int u(x)df_0(F(x))$.

It has been recently argued that in many cases it would be more appropriate to compare distributions by new "weighted" orderings of GL-type, that take into account deprivation, satisfaction and so on. These have the form

$$\int_0^p F_1^{-1}(\alpha)df_0(\alpha) \leq \int_0^p F_2^{-1}(\alpha)df_0(\alpha) \qquad \forall\, p\in[0,1) \tag{5}$$

for different choices of the f_0 function (e.g. star-shaped) (Chateauneuf and Moyes, 2005). Another consequence of Theorem 2, obtained when u_0 and f_0 are any two increasing functions, is:

The preference of F_2 over F_1, in a utilitarian sense, when the perception is f_0 and inequality aversion is at least as great as u_0, is equivalent to F_2 being preferred to F_1 in the extended GL sense defined by (5) by all decision-makers with utility u_0 and more inequality aversion than f_0.

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