A von Mises Markov random field model for the analysis of spatial circular data

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Abstract A von Mises Markov random field model is introduced for the analysis of spatial series of angles. Because the likelihood function of the model is unknown up to a normalizing constant, two inferential procedures are proposed for parameter estimation. The first one is based on the maximization of a pseudo-likelihood function and provides a computationally convenient, consistent, although inefficient estimator. The second one is based on the maximization of a Monte Carlo Markov Chain approximation of the likelihood and is more efficient than the pseudo-likelihood estimator, although computationally more expensive. The model is illustrated on a spatial series of sea currents directions.

Key words: circular spatial series, Markov Chain Monte Carlo, marine data, Markov field, pseudo-likelihood, von Mises

1 Introduction

Spatial series of angles arise when a circular variable is observed at a number of sites in a spatial domain and appear often in environmental, epidemiological and ecological studies. These data are often analyzed in a regression framework, exploiting a spatial linear predictor that includes the spatial coordinates of the observation sites and linking the expected response of a circular variable to the predictor through a suitable link function that maps the real line into the circle. Widely used is, for example, the von Mises regression model, where the mean parameter of the von Mises density is linked to the spatial predictor through an arctangent link function.

Inference in circular regression models is normally carried out by assuming that the observed angles are conditionally independent, given the linear predictor. This

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independence assumption, however, is often too restrictive in the analysis of spatial series. Only the large-scale spatial variation of the data is typically explained by a spatial predictor and regression residuals are often spatially autocorrelated, due to the small-scale variation of the data.

The literature on the spatial modeling of angles is limited. In a geostatistical context, [9] introduced the cosineogram as a measure of spatial autocorrelation between angles. In a Bayesian framework, [8] introduced a spatial autoregressive model for circular data by transforming and combining two spatial conditional autoregressive (CAR) processes.

Taking a likelihood-based approach to inference, a von Mises Markov random field (VM-MRF) model is proposed for the analysis of spatial circular data. The VM-MRF is a Markov random field whose coordinates are random variables with a circular support and whose finite-dimensional marginal distributions belong to an exponential family with low-dimensional sufficient statistics. Both the Markovian structure and the low-dimensionality of the model allow for an intuitively appealing interpretation of the parameters and facilitate the implementation of inferential procedures.

2 The von Mises Markov random field model

The VM-MRF is a spatial array of random variable Y_i, indexed by the sites i of a spatial domain S and defined on the circle, $y_i \in [-\pi, \pi)$. It is defined by the n-dimensional multivariate von Mises densities

$$f(\boldsymbol{y};\boldsymbol{\theta}) = \frac{\exp\{\kappa \boldsymbol{c}(\boldsymbol{y}-\boldsymbol{\mu}(\boldsymbol{\beta}))\mathbf{1} + \frac{\lambda}{2}\boldsymbol{s}\,(\boldsymbol{y}-\boldsymbol{\mu}(\boldsymbol{\beta}))^{\mathsf{T}}\,\mathbf{C}\;\boldsymbol{s}\,(\boldsymbol{y}-\boldsymbol{\mu}(\boldsymbol{\beta}))\}}{C(\kappa,\lambda)}, \qquad (1)$$

known up to the parameter $\boldsymbol{\theta} = (\boldsymbol{\beta}, \lambda, \kappa)$, where

$$\boldsymbol{c}(\boldsymbol{y}-\boldsymbol{\mu}(\boldsymbol{\beta}))^{\mathsf{T}} = (\cos(y_1-\mu_1(\boldsymbol{\beta}))\ldots\cos(y_n-\mu_n(\boldsymbol{\beta})))$$
$$\boldsymbol{s}(\boldsymbol{y}-\boldsymbol{\mu}(\boldsymbol{\beta}))^{\mathsf{T}} = (\sin(y_1-\mu_1(\boldsymbol{\beta}))\ldots\sin(y_n-\mu_n(\boldsymbol{\beta})))\,,$$

whereas $C(\kappa,\lambda)$ is the normalizing constant, $\mu_i(\boldsymbol{\beta})$ is a mean response linked to a linear predictor $\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta}$ trough a suitable link function mapping the real line to the circle and an available covariate profile $\boldsymbol{x}_i^{\mathsf{T}}$ that includes the spatial coordinates of the observation site, κ is a concentration parameter, λ is a spatial dependence parameter and, finally, $\boldsymbol{C}$ is a binary, symmetric adjaciency $n \times n$ matrix that specifies a spatial neighborhood structure between observation sites, i.e. each site $i \in S$ is associated with a neighborhood $N(i) \subset S$ and the generic element c_{ij} of $\boldsymbol{C}$ is equal to 1 if $j \in N(i)$ and 0 otherwise.

The likelihood function of a VM-MRF model is a special case of the multivariate von Mises density, introduced in [7] for bioinformatic studies and recently exploited in enviromental studies [5, 6]. For large values of the concentration parameter, this

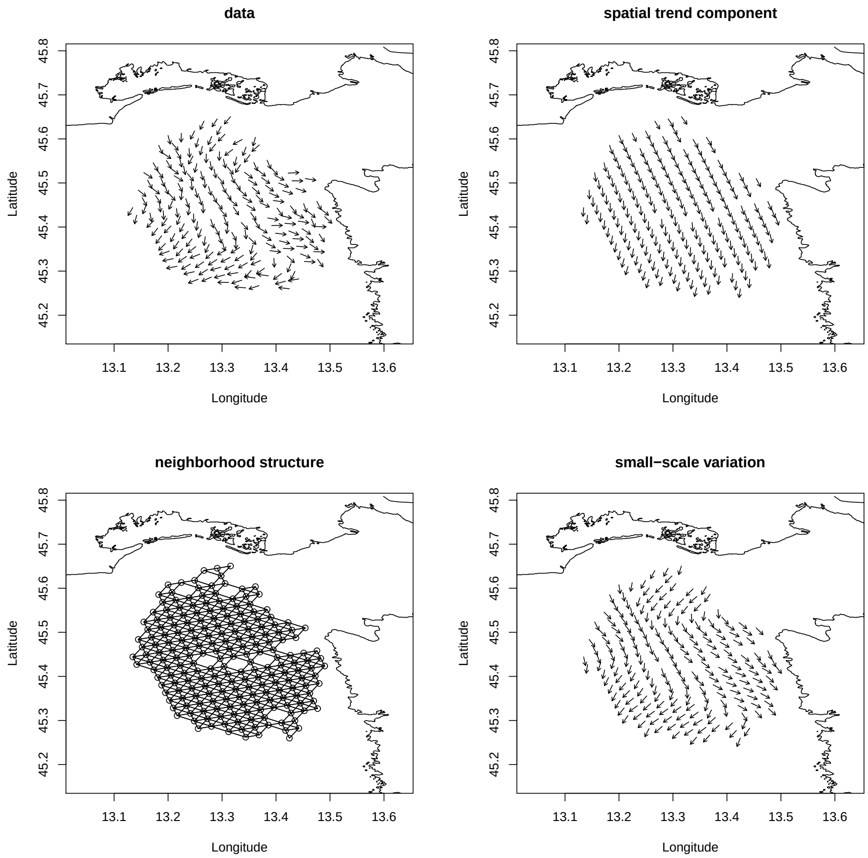

Fig. 1 Observed directions of sea currents (top left), spatial trend and small scale variation (top and bottom right) as predicted by a von Mises MRF through MCMC maximum likelihood; bottom left: the neighborhood structure used to define the von Mises MRF.

likelihood is well approximated by the likelihood of a Gaussian CAR process and maximum likelihood estimation is straightforward. Under low concentration levels, otherwise, the likelihood is known up to an intractable normalizing constant and direct likelihood maximization is not possible. Approximation methods such as pseudo-likelihood estimation [1] or Markov Chain Monte Carlo (MCMC) maximum likelihood [3] procedures are however relatively straightforward. Under a VM-MRF model, the univariate conditional distribution of each observation given the rest of the sample is von Mises, with parameters that depend only on the values taken by the field in the local neighborhood. As a result, maximization of the pseudo-likelihood reduces to a von Mises regression problem and the samples required by MCMC maximum likelihood can be generated by a simple Gibbs-sampling scheme. Moreover, the finite-dimensional marginal distributions (1) of the VM-MRF belong to an exponential family with low-dimensional sufficient statistics. This facilitates the numerical maximization of both the pseudo-likelihood and the MCMC-approximated likelihood function by standard Newton-Raphson algorithms.

3 Analysis of marine currents in the Adriatic sea

This paper was motivated by the analysis of sea current directions, provided by the NASCUM project, a program for the mapping of the surface currents in the Northen Adriatic Sea with high frequency (HF) radars. This area is affected by intense maritime traffic and accidental spillage episodes caused by ship navigation. Models of coastal marine circulation can therefore help to assess the environmental risk associated with these pollution sources. Figure 1 (top left-hand side) depicts the HF data, during a Bora episode. A VM-MRF model was fitted, exploiting a linear spatial trend as predictor and using an inverse tangent transformation as a link function. A queen's adjacency structure (Figure 1, bottom-left picture) was exploited to detect the small-scale spatial variation of the data. The results depicted in Figure 1 (top and bottom right-hand side) have been obtained by the maximization of a MCMC approximation to the likelihood function. The spatial trend component is given by the marginal expectation $\boldsymbol{\mu}(\hat{\boldsymbol{\beta}})$, while the small-scale variation component is given by the conditional expectation $\mathbb{E}(Y_i|\boldsymbol{y}_{N(i)};\hat{\boldsymbol{\theta}})$, where $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}, \hat{\kappa}, \hat{\lambda})$ are the MCMC maximum likelihood estimates, obtained from the data. It is worth noting that the decomposition of the data into a small scale and a large scale component depends on both the choice of a simple linear gradient and a queen's adjacency structure and, as a result, should be interpreted with care [4].

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