Asymptotic estimation of right and left kurtosis measures, with applications to finance

Anna Maria Fiori and Davide Beltrami

Abstract Inequality-based measures of right and left kurtosis have recently emerged as an effective alternative to the conventional fourth moment coefficient of kurtosis. In this contribution we show that the theory of L-statistics provides a convenient framework for the construction of empirical estimators for the new measures and the investigation of their asymptotic properties. Natural applications arise in financial contexts, in which the proposed estimators provide both a more robust and a more informative picture of the kurtosis risk embedded in market returns.

Key words: L-statistics, kurtosis curve, heavy tails, standard fourth moment.

1 Introduction

Consider a random variable X with finite first moment and define the conditional random variables:

$$D = X - \xi_{0.5} | X > \xi_{0.5}$$

$$S = \xi_{0.5} - X | X \leq \xi_{0.5}$$

which describe right and left excesses of X with respect to its median, $\xi_{0.5} = F^{-1}(0.5)$. Denote by δ_D the expectation of D and by δ_S the expectation of S.

As explained in [9] and [4], kurtosis increases if concentration (inequality) increases at either side of the median. Consequently, the *Gini indexes* of D:

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$$\phi_{F,D} = 2\int_0^1 [p - L_D(p)]\,dp \tag{1}$$

and S:

$$\phi_{F,S} = 2\int_0^1 [p - L_S(p)]\,dp \tag{2}$$

may be adopted as measures of right and left kurtosis, respectively. Here, L_D and L_S stand for the Lorenz curves (see, e.g., [1]) of D and S. In this framework, the conventional kurtosis coefficient (see, e.g. [6]):

$$\beta_2 = E\left(\frac{X-\mu}{\sigma}\right)^4 \tag{3}$$

no longer plays a central role, nor is the normal distribution needed as a reference model. Based on (1) and (2), kurtosis is easily defined and interpreted in both symmetric and asymmetric contexts. In addition, a partial ordering of distributions by right/left kurtosis (which is naturally imposed by nested Lorenz curves [4]) has recently met the interest of the financial community [8].

2 Empirical estimators of right and left kurtosis measures

Using the theory of L-statistics, we construct empirical estimators for (1) and (2) with appealing asymptotic properties. Consider independent random variables $X_1, X_2, \ldots, X_n$ (n even), each having the same cdf F as X, and denote by $X_{n1}, X_{n1}, \ldots, X_{nn}$ the corresponding order statistics. The right kurtosis measure (1) may be estimated by the following ratio of L-statistics:

$$\phi_{n,D} = \frac{\frac{2}{n}\sum_{i=n/2+1}^{n}\left(\frac{4i-2}{n}-3\right)X_{ni}}{\frac{2}{n}\sum_{i=n/2+1}^{n} X_{ni} - F_n^{-1}(0.5)} \tag{4}$$

while an empirical estimator of the left kurtosis measure (2) is given by:

$$\phi_{n,L} = \frac{\frac{2}{n}\sum_{i=1}^{n/2}\left(\frac{4i-2}{n}-1\right)X_{ni}}{F_n^{-1}(0.5) - \frac{2}{n}\sum_{i=1}^{n/2} X_{ni}} \tag{5}$$

Theorem 1. *Suppose that F has a positive and continuous derivative f in a neighborhood of $F^{-1}(0.5)$ and $E|X|^{2+r} < \infty$ for some $r > 0$. Under these assumptions, it can be shown that:*

$$\sqrt{n}\left(\phi_{n,D} - \phi_{F,D}\right) \to_d N\left(0; \sigma^2_{F,D}\right)$$

where:

$$\sigma^2_{F,D} = \frac{1}{\delta^2_D} Var_F \{h_D(F;X)\} < \infty$$

and:

$$h_D(F;X) = -2\int_{F^{-1}(0.5)}^{\infty} [I(X \leq x) - F(x)] \cdot [4F(x) - 3 + \phi_{F,D}]\, dx + \phi_{F,D} \frac{0.5 - I\left[X \leq F^{-1}(0.5)\right]}{f\left[F^{-1}(0.5)\right]}$$

An analogous result holds for the left kurtosis estimator (5)[1].

While the sampling variance of the conventional kurtosis estimator:

$$b_2 = \frac{\frac{1}{n}\sum (X_i - \bar{X})^4}{\left\{\frac{1}{n}\sum (X_i - \bar{X})^2\right\}^2} \tag{6}$$

is related to the population moment of order eight, the new measures (1) and (2) can be conveniently estimated under the milder requirement that second (or slightly higher) moments are finite. Monte Carlo simulations from increasingly heavy tailed parent distributions have been carried out in [5], showing a faster convergence of the sampling distributions of (4) and (5) to the limiting normal model. Conversely, the sampling distribution of (6) is affected by persistent skewness even for large n.

3 Application

For purposes of financial applications, the interplay of kurtosis and clustering in the volatility dynamics needs to be considered. Consequently, we first fit ARMA-GARCH models to several series of daily logarithmic returns by QML techniques. Right and left kurtosis measures are then estimated on standardized residuals by (4) and (5), respectively. Approximate 95% confidence intervals are computed, using the results of Theorem 1 and consistent estimators for the asymptotic variances. A few examples of our applications are summarized in Table 1. As expected, conventional kurtosis (6) is always significantly larger than its reference value of 3 at the normal distribution, even in periods of moderate market movements. Conversely, the asymptotic confidence intervals for either $\phi_{F,D}$ or $\phi_{F,S}$ frequently contain the reference value of 0.4142 (which is the common value of right and left kurtosis at the normal distribution). After removing the two largest losses and gains from the sample, the new measures are nearly unchanged, while the conventional kurtosis estimator (6) and its approximate 95% confidence interval (see, e.g. [3]) become compatible with a normal distribution.

[1] A detailed proof of these results is beyond the scope of this contribution and will be provided in a separate paper.

Table 1 A few examples of sample kurtosis measures and their approximate 95% confidence intervals (in brackets). The full sample runs from 07-01-2003 to 29-01-2007 and includes 1000 ARMA-GARCH residual returns. The reduced sample, of size 996, is obtained by removing the two largest losses and gains. * indicates that the sample measure is significantly different from its reference value at the normal model.

Return series	Conventional kurtosis (6)	Right kurtosis (4)	Left kurtosis (5)
GBP/JPY exch. rate (*full sample*)	4.623^* $\{3.289; 5.957\}$	0.423 $\{0.403; 0.443\}$	0.472^* $\{0.450; 0.495\}$
GBP/JPY exch. rate (*reduced sample*)	3.377 $\{2.982; 3.773\}$	0.419 $\{0.399; 0.439\}$	0.464^* $\{0.443; 0.485\}$
Nikkei 225 index (*full sample*)	3.742^* $\{3.103; 4.380\}$	0.413 $\{0.393; 0.432\}$	0.464^* $\{0.441; 0.487\}$
Nikkei 225 index (*reduced sample*)	3.176 $\{2.856; 3.495\}$	0.410 $\{0.391; 0.429\}$	0.457^* $\{0.436; 0.479\}$

These findings suggest that leptokurtosis may have been accepted too readily as a generalized feature of market return distributions, as also argued in [7] and [2]. Looking at right and left kurtosis separately is more revealing, since departures from normality can be associated with either the left tail (which mainly represents losses) or the right tail (gains). The new measures of right and left kurtosis are thus likely to provide both a more robust and a more sophisticated picture of the kurtosis risk embedded in market return data.

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