Handling weak dependence structures with copulas

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Abstract We provide a method to construct a class of n–copulas $\mathbf{C}$ defined as $\mathbf{C}(\mathbf{u}) = \mathbf{D}(u_1, \ldots, u_{n-1})u_n + \mathbf{A}(u_1, \ldots, u_{n-1})f(u_n)$. These copulas are obtained from a $(n-1)$–copula $\mathbf{D}$ and some suitable auxiliary functions $\mathbf{A}$ and f. Members of this class have been fully characterized in terms of properties of the auxiliary functions. The proposed copulas may model some weak dependence and a non–exchangeable behavior among the components of a random vector. Moreover, they can be easily simulated and fitted to real data.

Key words: Copula, Dependence, Farlie–Gumbel–Morgenstern distribution.

1 Introduction

Motivated by the recent interest about dependence and related concepts for the description of stochastic behavior of correlated random phenomena in economics, finance and geoscience, a number of papers has been recently devoted to the construction of higher dimensional copulas. See, for instance, [4, 5, 7, 8]. Moreover, there are several examples in which it is essential to consider weak dependence structures instead of simple independence. This is in particular related to some of the most popular conditions used by econometricians to transcribe the notion of fading memory.

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Here we are interested on a special construction for copulas, which is based on the modification of a known copula by adding to its expression a factor term in order to allow more flexibility in the dependence structure. Constructions of this type are sometimes called *perturbations of a copula* [2] and have been implicitly used in the literature several times; e.g., see the Farlie–Gumbel–Morgenstern distribution (hereafter, FGM) and its various modifications [1].

Specifically, we extend an $(n-1)$–copula $\mathbf{D}$ to a n–copula $\mathbf{C}$ such that $\mathbf{D}$ is a multivariate margin of $\mathbf{C}$ in most cases. Methods of this type are particularly useful when one knows the behavior of a $(n-1)$ random vector, say $\mathbf{X}' = (X_1, X_2, \dots, X_{n-1})$, but there is small evidence about the relations between $\mathbf{X}'$ and another random variable X_n that should be included into the model. This approach has some other features of interest; i.e., dependent random samples can be easily simulated (via conditional distribution method) and copula parameters estimated by using maximum pseudo–likelihood techniques.

The described method has been investigated in detail in [3], where an application to hydrological data is also given.

2 The model

Let $\mathbb{I} = [0,1]$. Given $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{I}^n$, we denote $\mathbf{u}' = (u_1, \dots, u_{n-1})$. For $n \geq 3$ and a given $(n-1)$–copula $\mathbf{D}$, we aim at considering n–copulas that can be expressed in the form

$$\mathbf{C}(\mathbf{u}) = \mathbf{D}(\mathbf{u}')\, u_n + \mathbf{A}(\mathbf{u}')\, f(u_n) \tag{1}$$

for suitable functions $\mathbf{A}\colon \mathbb{I}^{n-1} \to \mathbb{R}$ and $f\colon \mathbb{I} \to \mathbb{R}$.

Notice that, if $f(t) = t(1-t)$ for every $t \in \mathbb{I}$, then copulas of type (1) have been considered in [9], where it is also shown that they include the FGM family of multivariate copulas. The following result characterizes the n-copulas of type (1).

Proposition 1. *Let* $\mathbf{D}$ *be an* $(n-1)$*–copula. If* $\mathbf{A}\colon \mathbb{I}^{n-1} \to \mathbb{R}$ *and* $f\colon \mathbb{I} \to \mathbb{R}$ *are two non–zero functions, then* $\mathbf{C}$ *defined by* (1) *is an n–copula if, and only if,* $\mathbf{A}$ *and* f *satisfy the following conditions:*

(A1) $\mathbf{A}(\mathbf{1}) = 0$ *and* $\mathbf{A}(\mathbf{u}') = 0$ *for all* $\mathbf{u}' \in \mathbb{I}^{n-1}$ *having at least one component equal to* 0*;*

(A2) $f(0) = 0$ *and* $f(1) = 0$ *or, otherwise,* $\mathbf{A}(\mathbf{u}') = 0$ *for all* $\mathbf{u}' \in \mathbb{I}^{n-1}$ *having* $n-2$ *components equal to* 1*;*

(A3) f *is absolutely continuous and* $\alpha_{\mathbf{D},\mathbf{A}} \leq f'(t) \leq \beta_{\mathbf{D},\mathbf{A}}$ *for almost every* $t \in \mathbb{I}$*, where* $\alpha_{\mathbf{D},\mathbf{A}}$ *and* $\beta_{\mathbf{D},\mathbf{A}}$ *are provided in [3].*

Here we provide some examples that illustrate the usefulness of previous result.

Example 1. Let $\mathbf{D} = \Pi_{n-1}$ the independence copula and let $f\colon \mathbb{I} \to \mathbb{R}$. Let $\mathbf{A}\colon \mathbb{I}^{n-1} \to \mathbb{R}$ be the function defined by $\mathbf{A}(\mathbf{u}') = \prod_{i=1}^{n-1} u_i(1-u_i)$. Thus, $\mathbf{C}\colon \mathbb{I}^n \to \mathbb{I}$ given by

$$\mathbf{C}(\mathbf{u}) = \Pi_n(\mathbf{u}) + \prod_{i=1}^{n-1} u_i\,(1-u_i)\,f(u_n), \quad \mathbf{u} \in \mathbb{I}^n,$$

is an n–copula if, and only if, f is a 1–Lipschitz function satisfying (A3). This model can be used when all the variables of interest are pairwise independent, but the model is not globally independent.

Example 2. Let $\mathbf{D}$ be an $(n-1)$–copula and let f be the absolutely continuous function defined on $\mathbb{I}$ by $f(t) = t(1-t)(2+t)$. Let $\mathbf{A}\colon \mathbb{I}^{n-1} \to \mathbb{R}$ be a function satisfying conditions (A1) and (A2). It follows that

$$\mathbf{C}(\mathbf{u}) = \mathbf{D}(\mathbf{u}')\,u_n + \mathbf{A}(\mathbf{u}')\,u_n\,(1-u_n)\,(2+u_n) \tag{2}$$

is an n–copula if, and only if, $-V_{\mathbf{D}}(J')/2 \le V_{\mathbf{A}}(J') \le V_{\mathbf{D}}(J')/3$ for every $(n-1)$–box J' in $\mathbb{I}^{n-1}$. Copulas of type (2) belong to the family of n–copulas with cubic section in one variable discussed in [10].

Example 3. Let us consider $\mathbf{D} = M_{n-1}$ the comonotonicity copula, and let $\mathbf{A}\colon \mathbb{I}^{n-1} \to \mathbb{R}$ be given by

$$\mathbf{A}(\mathbf{u}') = \begin{cases} 1 - M_{n-1}(\mathbf{u}'), & \mathbf{u}' \in [\frac{1}{2}, 1]^{n-1}, \\ M_{n-1}(\mathbf{u}'), & \text{otherwise.} \end{cases}$$

Let $f\colon \mathbb{I} \to \mathbb{R}$ be a function such that $f(0) = f(1) = 0$. Then

$$\mathbf{C}(\mathbf{u}) = \begin{cases} M_{n-1}(\mathbf{u}')(u_n - f(u_n)) + f(u_n), & \mathbf{u}' \in [\frac{1}{2}, 1]^{n-1}, \\ M_{n-1}(\mathbf{u}')(u_n + f(u_n)), & \text{otherwise} \end{cases}$$

is an n–copula if, and only if, f is a 1–Lipschitz function. Notice that such a $\mathbf{C}$ is not absolutely continuous.

3 The model in practice

Here, we present some features of the obtained model that could be useful from a practical viewpoint.

Probabilistic interpretation. Intuitively, copulas of type (1) have been obtained from a basis copula, namely $\mathbf{D}(\mathbf{u}')\,u_n$, which is modified by means of an additive term, namely $\mathbf{A}(\mathbf{u}')\,f(u_n)$, in order to ensure that $\mathbf{C}$ describes a wider range of weak dependence than the basis copula. In fact, if $\mathbf{X}$ were a random vector whose copula is $\mathbf{D}(\mathbf{u}')\,u_n$, then X_n would be independent from the other variable. But, in such a model, the modification $\mathbf{A}(\mathbf{u}')\,f(u_n)$ includes artificially a link between X_n and the other components of $\mathbf{X}$.

Richness of the method. The present construction principle allows to obtain families of copulas that are more general than the classes previously considered (see for instance [1, 9, 10]). In particular, we may obtain copulas with a singular component. This feature is especially used when one wants to model loss random variables in financial context and would like to allow for joint defaults.

Asymmetry. As a special feature, copulas of type (1) may model a non–exchangeable behavior among the variables of interest. This flexibility is not possible with some of the widespread families of copulas, like Archimedean and Gaussian copulas.

Tail behavior. Copulas of type (1) may describe some joint heavy–tail behavior of the variables under consideration. In particular, contrarily to most of the families of copulas derived from FGM models, they may add some positive probability mass to the tail of the distribution.

Sampling procedure. Usually, the derivatives of a copula $\mathbf{C}$ of type (1) can be easily calculated when the derivatives of $\mathbf{A}$ and $\mathbf{D}$ are computationally manageable. Thanks to this fact, in order to simulate a random sample from $\mathbf{C}$, the conditional distribution method [8] can be applied. Some simulations from copulas of this type appeared in [3].

Fitting procedure. The density of an absolutely continuous copula $\mathbf{C}$ of type (1) can be easily calculated when the density of $\mathbf{A}$ and $\mathbf{D}$ are computationally manageable. Thanks to this fact, fitting procedures based on maximum pseudo–likelihood estimation can be adopted for such a model.

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