# Identifiability of Discrete Graphical Models with Hidden Variables

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**Abstract** We define a space of identifiability problems in causal Bayesian networks and concentrate on two of them. The first problem involves the generic identifiability of all parameters with restrictions on the state space of the variables. We present a technique that, given an arbitrary directed graphical model with a single hidden variable, modifies the model in such a way that we can apply Kruskal's theorem and solve the first identifiability problem. The second problem involves the global identifiability of the causal effect of a set $T$ of variables on a set $S$ of variables. Pearl's do-calculus solves the second identifiability problem.

**Key words:** Causal Bayesian networks, Semi-Markovian models, Intervention, Identifiability, Unidentifiability

## 1 Two Settings for Identifiability

Markovian models are popular graphical models for encoding distributional and causal relationships. A *Markovian model* consists of an acyclic directed graph (DAG) $G$ over a set of variables $V = \{V_1, \ldots, V_n\}$, called a *causal graph*, and a probability distribution over $V$, which satisfies two constraints: each variable in the graph is independent of all its non-descendants given its direct parents, and the directed

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edges in $G$ represent direct causal influences. A Markovian model for which only the first constraint holds is called a *Bayesian network*. This explains why Markovian models are also called *causal Bayesian networks*.

The *chain rule for Bayesian networks* states that the joint probability function $P(v) = P(v_1, \ldots, v_n)$ can be factorized as

$$P(v) = \prod_{V_i \in V} P(v_i | pa(V_i)) \tag{1}$$

The simplest kind of *intervention* [4] is fixing a subset of $V$, $T$, to some constants $t$, denoted by $do(T = t)$ or just $do(t)$, and then the post-intervention distribution $P_T(V)(T = t, V = v) = P_t(v)$ is compatible with the *excision semantics* and given by:

$$P_t(v) = \begin{cases} \prod_{V_i \in V \setminus T} P(v_i | pa(V_i)) & v \text{ consistent with } t \\ 0 & v \text{ inconsistent with } t \end{cases} \tag{2}$$

Let $N$ and $U$ stand for the sets of observable (observed) and unobservable (hidden) variables in graph $G$, i.e., $N$ and $U$ partition $V$. The observed probability distribution is:

$$P(n) = \sum_U \prod_{V_i \in N} P(v_i | pa(V_i)) \prod_{V_j \in U} P(v_j | pa(V_j)) \tag{3}$$

One can define a space of identifiability problems based on equation (3). We concentrate on three dimensions of this space: identifiability of all parameters or only some of them, identifiability of parameters in their whole range (*global* identifiability) or with the exception of some subspace of measure zero (*generic* identifiability), and identifiability with restrictions on the cardinality of the state space of variables or without them. We call *identifiability_1* the generic identifiability of all the probabilities in (3) with appropriate bounds on the state spaces of variables, and *identifiability_2* the global identifiability with no bounds on the state space of variables of the *causal effect* $P_t(s)$, given by:

$$P_t(s) = \begin{cases} \sum_{V_l \in (N \setminus S) \setminus T} \sum_U \prod_{V_i \in N \setminus T} P(v_i | pa(V_i)) \times & \\ \prod_{V_j \in U} P(v_j | pa(V_j)) & s \text{ consistent with } t \\ 0 & s \text{ inconsistent with } t \end{cases} \tag{4}$$

## 2 Kruskal Theorem and Its Use to Solve Identifiability_1

Kruskal's theorem applies to a simple latent class model, in which three observed variables are independent when conditioned on a single hidden one. We outline a technique that, given an arbitrary directed graphical model with a single hidden variables, modifies the model in such a way that we can apply Kruskal's theorem. The technique, which we have been developing and generalizes the one in [1], is based on the following operations:

- Clump several variables (all hidden or all observed) into a single one, with larger state space.
- Condition on the state of an observed variable.
- Marginalize over an observed variable (making it hidden).

Each of these can be done multiple times, and in combination with one another. The goal in applying these modifications is always to produce a model to which Kruskal's theorem applies, so one needs to use them so that:

- At least 3 observed variables remain, which are independent when conditioned on the hidden variable.
- The resulting hidden state spaces are "not too large" relative to observed ones. (Letting $a, b, c$, and $q$ be the sizes of the state spaces of the observed and hidden variables, in order, then $\min(a,q) + \min(b,q) + \min(c,q) \geq 2q + 2$.)
- Parameters for the resulting model are easily related to those of the original one.

It is easy to show by a counting argument that all Bayesian networks of four nodes in which there is at least one edge between children of the hidden variable are not identifiable. An example of such a network is in Figure 1(a).

The causal Bayesian network of Figure 1(b) is identifiable by conditioning on variable 2, applying Kruskal's theorem on the resulting network of three observed nodes and inverting the resulting conditional probability tables.

## 3 Using the Do-calculus to Solve Identifiability_2

The do-calculus consists of three rules that allow the replacement of interventions with observations in modified graphs [4]. Let $X$, $Y$, $Z$ be arbitrary disjoint sets of nodes in a causal graph $G$. We denote by $G_{\overline{X}}$ the graph obtained by deleting from $G$ all edges pointing to nodes in $X$ and by $G_{\underline{X}}$ the graph obtained by deleting from $G$ all edges emerging from nodes in $X$. To represent the deletion of both incoming and outgoing edges, we use the notation $G_{\overline{X}\underline{Z}}$.

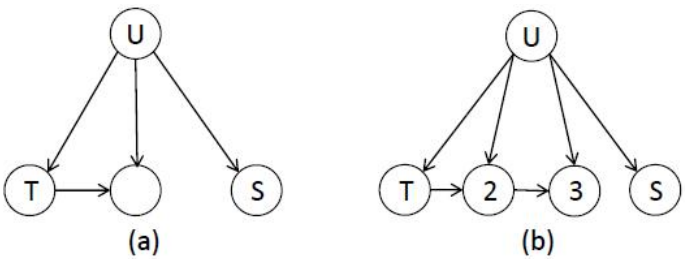


**Fig. 1** The causal Bayesian network (whose graph is) (a) is not identifiable_1, but the causal effect $P_t(s)$ is identifiable_2. The causal Bayesian network (b) is identifiable_1, but $P_t(s)$ is not identifiable_2.

(*Rules of Do-Calculus*) Let $G$ be the DAG of a causal Bayesian network, and let $P(.)$ stand for its probability distribution. For any disjoint subsets of variables $X$,$Y$,$Z$, and $W$ we have the following rules.

Rule 1 (Insertion/deletion of observations)

$$P_x(y|z,w) = P_x(y|w) \;\; if (Y \perp Z|X,W)_{G_{\overline{X}}} \tag{5}$$

Rule 2 (Action/observation exchange)

$$P_{xz}(y|w) = P_x(y|z,w) \;\; if (Y \perp Z|X,W)_{G_{\overline{X}\underline{Z}}} \tag{6}$$

Rule 3 (Insertion/deletion of actions)

$$P_{xz}(y|w) = P_x(y|w) \;\; if (Y \perp Z|X,W)_{G_{\overline{X},\overline{Z(W)}}} \tag{7}$$

where $Z(W)$ is the set of $Z$-nodes that are not ancestors of any $W$-node in $G_{\overline{X}}$.

It was shown that the do-calculus is sound and complete [2] for the identifiability_2 problem, i.e., a causal effect is identifiable_2 if and only if the quantity $P_t(s)$ can be tranformed into a formula that includes only observable quantities (i.e., quantities derivable from $P(N)$) by using the rules of the do-calculus and standard probability manipulations. To show that a causal effect is unidentifiable, it is however more convenient to use the algorithm of Tian [6], which was also shown to be sound and complete [5, 3]. For example, $P_T(S)$ is identifiable_2 in the graph of Figure 1(a), because $P_T(S) = P(S)$; in other words, $T$ has no causal effect on $S$. This can also be shown by applying rule 3 (equation (7)), with X = $\{\}, Y = S, Z = T, W = \{\}$,; consequently, $Z(W) = T$, and $G_{\overline{X},\overline{Z(W)}}$ is the graph of Figure 1(a) without the edge $(U,T)$. $P_T(S)$ is not identifiable_2 in the graph of Figure 1(b).

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