Large sample properties of Gibbs-type priors

Pierpaolo De Blasi, Antonio Lijoi and Igor Prünster

Abstract In this paper we concisely summarize some recent findings that can be found in [1] and concern large sample properties of Gibbs-type priors. We shall specifically focus on consistency according to the frequentist approach which postulates the existence of a " true " distribution P_0 that generates the data. We show that the asymptotic behaviour of the posterior is completely determined by the probability of obtaining a new distinct observation. Exploiting the predictive structure of Gibbs-type priors, we are able to establish that consistency holds essentially always for discrete P_0, whereas inconsistency may occur for diffuse P_0. Such findings are further illustrated by means of three specific priors admitting closed form expressions and exhibiting a wide range of asymptotic behaviours.

Key words: Asymptotics, Bayesian nonparametrics, Gibbs-type priors

1 Gibbs-type priors

In this paper we sketch results that are extensively presented and proved in [1] about the asymptotic posterior behaviour of Gibbs-type priors, a class of discrete nonparametric priors recently introduced in [5]. Gibbs-type priors can be defined through the system of predictive distributions they induce. To this end, let $(X_n)_{n\geq 1}$ be an (ideally) infinite sequence of observations, with each X_i taking values in a complete and separable metric space $\mathbb{X}$. Moreover, $\mathbf{P}_{\mathbb{X}}$ is the set of all probability measures on $\mathbb{X}$ endowed with the topology of weak convergence. In the most commonly employed Bayesian models $(X_n)_{n\geq 1}$ is assumed to be *exchangeable* which means there exists a

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probability distribution Q on $\mathbf{P}_{\mathbb{X}}$ such that $X_i|\tilde{p} \stackrel{\text{iid}}{\sim} \tilde{p}, \quad \tilde{p} \sim Q$. Hence, $\tilde{p}$ is a random probability measure on $\mathbb{X}$ whose probability distribution Q is also termed *de Finetti measure* and acts as a prior for Bayesian inference. Given a sample $(X_1,\dots,X_n)$, the predictive distribution coincides with the posterior expected value of $\tilde{p}$, that is

$$\mathrm{pr}(X_{n+1} \in \cdot\, | X_1,\dots,X_n) = \int_{\mathbf{P}_{\mathbb{X}}} p(\,\cdot\,)\, Q(\mathrm{d}p | X_1,\dots,X_n). \tag{1}$$

A prior Q that selects, almost surely, discrete distributions is said discrete and, in this case, a sample $(X_1,\dots,X_n)$ will feature ties with positive probability: $X_1^*,\dots,X_k^*$ denote the $k \le n$ distinct observations and $n_1,\dots,n_k$ their frequencies for which $\sum_{i=1}^k n_i = n$. Gibbs-type priors are discrete and characterized by predictive distributions (1) of the form

$$\mathrm{pr}(X_{n+1} \in \cdot\, | X_1,\dots,X_n) = \frac{V_{n+1,k+1}}{V_{n,k}} P^*(\,\cdot\,) + \frac{V_{n+1,k}}{V_{n,k}} \sum_{i=1}^k (n_i - \sigma)\delta_{X_i^*}(\,\cdot\,), \tag{2}$$

where $\sigma \in (-\infty, 1)$, $P^*(\mathrm{d}x) := E[\tilde{p}(\mathrm{d}x)]$ is a diffuse probability measure representing the prior guess at the shape of $\tilde{p}$ and $\{V_{n,k} : k = 1,\dots,n; n \ge 1\}$ is a set of non-negative weights satisfying the recursion

$$V_{n,k} = (n - \sigma k)V_{n+1,k} + V_{n+1,k+1}. \tag{3}$$

Therefore, Gibbs-type priors are characterized by predictive distributions, which are a linear combination of the prior guess and a weighted version of the empirical measure. The most widely known priors within this class are the Dirichlet process [3] and the two-parameter Poisson-Dirichlet process [8].

2 Consistency results

We address posterior consistency according to the " what if " approach of [2], which consists in assuming that the data $(X_n)_{n\ge1}$ are independent and identically distributed from some " true " $P_0 \in \mathbf{P}_X$ and in verifying whether the posterior distribution $Q(\,\cdot\,|X_1,\dots,X_n)$ accumulates in any neighborhood of P_0, under a suitable topology. Since Gibbs-type priors are defined on $\mathbf{P}_{\mathbb{X}}$ and are discrete, the appropriate notion of convergence is convergence in the weak topology. Therefore, we aim at establishing whether $Q(A_\varepsilon|X_1,\dots,X_n) \to 1$, a.s.-$P_0^\infty$, as $n \to \infty$ and for any $\varepsilon > 0$, where A_ε denotes a weak neighborhood of some $P' \in \mathbf{P}_X$ of radius ε and P_0^∞ is the infinite product measure. Clearly, consistency corresponds to $P' = P_0$.

We prove a general structural result on Gibbs-type priors showing that the posterior distribution converges to a point mass at the weak limit, in an almost sure sense, of the predictive distribution (2). To this aim, we shall assume that the probability of recording a new distinct observation at step $n+1$

$$\frac{V_{n+1,\kappa_n+1}}{V_{n,\kappa_n}} \quad \text{converges} \qquad \text{a.s.-}P_0^\infty \tag{H}$$

as $n \to \infty$, and the limit is identified by some constant $\alpha \in [0,1]$. We use the notation κ_n in order to make explicit the dependence on n of the number of distinct observations in a sample of size n. Different choices of P_0 yield different limiting behaviours for κ_n: if P_0 is discrete with N point masses, then $P_0^\infty(\lim_n n^{-1}\kappa_n = 0) = 1$ even when $N = \infty$; if P_0 is diffuse, $P_0^\infty(\kappa_n = n) = 1$ for any $n \geq 1$. The following theorem shows that (H) is actually sufficient to establish weak convergence at a certain P' that is explicitly identified. The key ingredient for the proof is represented by an upper bound on the posterior variance $Var[\tilde{p}(A)\,|\,X_1,\dots,X_n]$, which is of independent interest. See [1].

Theorem 1. *Let $\tilde{p}$ be a Gibbs-type prior with prior guess $P^* = E[\tilde{p}]$, whose support coincides with $\mathbb{X}$, and assume condition* (H) *holds true. If $(X_i)_{i\geq 1}$ is a sequence of independent and identically distributed random elements from P_0 then the posterior converges weakly, a.s.-P_0^∞, to a point mass at $\alpha P^*(\cdot) + (1-\alpha)P_0(\cdot)$.*

According to Theorem 1, weak consistency is achieved in the trivial case of $P^* = P_0$, which will be excluded henceforth, and when $\alpha = 0$: therefore, it is sufficient to check whether the probability of obtaining a new observation, given previously recorded data, converges to 0, a.s.-P_0^∞. One might also wonder whether there are circumstances leading to $\alpha = 1$, which corresponds to the posterior concentrating around the prior guess P^*, a situation we refer to as "total" inconsistency.

Since a few particular cases of Gibbs-type priors with $\sigma \in (0,1)$ have already been considered in [7] and [6], attention is focused on the case of $\sigma \in (-\infty, 0)$ for which nothing is known to date. We recall here that, if $\sigma < 0$, Q is a mixture of Poisson-Dirichlet processes with parameters $(\sigma, k|\sigma|)$ and the mixing distribution for k, say π, is supported by the positive integers. Since in the case of negative σ the two-parameter model coincides with a x-variate symmetric Dirichlet distribution, one can describe such Gibbs-type priors as

$$\begin{aligned} (\tilde{p}_1,\dots,\tilde{p}_k) &\sim \text{Dirichlet}(|\sigma|,\dots,|\sigma|) \\ k &\sim \pi(\cdot) \end{aligned} \tag{4}$$

We shall restrict attention to Gibbs-type priors whose realizations are discrete distributions whose support has a cardinality that cannot be bounded by any positive constant, almost surely. This is the same as assuming that the support of π in (4) is $\mathbb{N}$. Note that for the " parametric " case of $\sigma < 0$ and π supported by a finite subset of $\mathbb{N}$ one immediately has consistency for any P_0 in its support by the results of [4]. Theorem 2 gives neat sufficient conditions for consistency in terms of the tail behaviour of the mixing distribution π on the positive integers $\mathbb{N}$ in (4).

Theorem 2. *Let $\tilde{p}$ be a Gibbs-type prior with parameter $\sigma < 0$, mixing measure π and prior guess P^* whose support coincides with $\mathbb{X}$. Then the posterior is consistent*

(i) *at any discrete P_0 if for sufficiently large x*

$$\pi(x+1)/\pi(x) \leq 1; \tag{T1}$$

(ii) *at any diffuse P_0 if for sufficiently large x and for some $M < \infty$*

$$\pi(x+1)/\pi(x) \leq M/x. \tag{T2}$$

Note that condition (T1) is an extremely mild assumption on the regularity of the tail of the mixing π: it requires $x \mapsto \pi(x)$ to be ultimately decreasing, a condition met by the commonly used probability measures on $\mathbb{N}$. On the other hand, condition (T2) requires the tail of π to be sufficiently light. This is indeed a binding condition and it is particularly interesting to note that such a condition is also close to being necessary. In [1], three different Gibbs-type priors with $\sigma = -1$ are considered, each prior characterized by a specific choice of the mixing distribution π. These examples show that, according as to heaviness of the tails of π, the value of α in Theorem 1 may actually span the whole interval $[0,1]$, from situations where consistency holds true ($\alpha = 0$) to cases where " total " inconsistency occurs ($\alpha = 1$). In particular, the heavier the tail of π and the larger α, i.e. the lighter is the weight assigned to the "true" P_0 in the limiting distribution identified in Theorem 1. The first prior is characterized by a heavy-tailed mixing distribution π, which does not admit a finite expected value: condition (T2) is not met and it turns out that $\alpha = 1$ so that the posterior concentrates around the prior guess P^* (" total " inconsistency). The second specific prior, where the mixing π has light tails that satisfy (T2) in Theorem 2, results in a consistent asymptotic behaviour. In the third case α takes values over the whole unit interval $[0,1]$ according to a parameter that determines the heaviness of the tail of π.

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