Limited Information Estimation Methods for Paired Comparison Data

Manuela Cattelan

Abstract Paired comparison data are very common in psychometric experiments where a certain number of subjects express their preferences for each couple of a set of items compared pairwise. Traditional models for the analysis of paired comparison data were fitted assuming independence among all comparisons, but this is unrealistic. The difficulties in maximum likelihood estimation of models accounting for dependence have been overcome by means of alternative estimating techniques. We propose an optimal combination of estimating equations which requires only bivariate marginal distributions as an alternative to the limited information estimation method proposed in the psychometric literature.

Key words: limited information estimation, optimal estimating equations, paired comparison data

1 Introduction

Paired comparison data consist in the comparison of a set of items in couples. This type of data arise in many areas including sensory testing, genetics, sport tournaments and animal behavior analysis. In psychometrics they are very common because it is easier for people to compare two objects at a time than ranking a list of items. The traditional model used in psychometric analysis dates back to Thurstone [6], but the difficulties in the estimation of the model induced psychometricians to fit the model as if the data were independent. The other traditional model for the analysis of paired comparison data, the Bradley-Terry model [1], also assumes that comparisons are independent. This assumption is unrealistic because it implies that the comparisons made by the same person involving common objects are indepen-

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dent. Here we consider two methods that can be employed for fitting models for paired comparisons data that account for dependence.

2 Correlated Paired Comparison Data

Psychometricians are interested in understanding the relationship among a set of stimuli compared in couples. Let n denote the number of items or stimuli compared in couples and S be the number of subjects. People express their preferences for every pair of items, so everyone provides $N = n(n-1)/2$ comparisons. Let Y_{sij} denote the result of the comparison between items i and j, $i < j = 1, \dots, n$, made by subject s. $Y_{sij} = 1$ if subject s prefers item i to j and it is 0 otherwise. The original Thurstone [6] model assumes that the stimuli compared follow a normal distribution $T \sim N(\mu, \Sigma_T)$, with mean $\mu = (\mu_1, \dots, \mu_n)$ and covariance matrix Σ_T. Hence a preference for i is expressed by subject s, *i.e.* $Y_{sij} = 1$, iff $T_{si} > T_{sj}$, that is when $Z_{sij} = T_{si} - T_{sj} > 0$. Let $Z_s = (Z_{s12}, \dots, Z_{sn-1n})$ be the vector of all latent continuous random variables pertaining to subject s, then Takane [5] suggests the extension

$$Z_s = AT + \varepsilon,$$

where $\varepsilon = (\varepsilon_{s12}, \dots, \varepsilon_{sn-1n})$ is a vector of pair specific errors with zero mean and covariance Ω, independent of T and of $\varepsilon_{s'}$ for any other subject $s' \neq s$. A is the matrix of paired comparisons with rows corresponding to the paired comparisons and columns to the items. For example, if the paired comparisons $(1,2)$, $(1,3)$ and $(2,3)$ are performed by s the paired comparisons matrix is

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Hence Z_s follows a multivariate normal distribution with mean $A\mu$ and covariance $\Sigma_Z = A\Sigma_T A^T + \Omega$. This specification allows for correlation between comparisons performed by the same person involving a common object.

3 Estimation

3.1 Limited Information Estimation

The Thurstone model is not straightforward to estimate. Let $D = \mathrm{diag}(\Sigma_Z)^{-1/2}$ and $Z_s^* = D(Z_s - A\mu)$ be the standardized latent variable with mean 0 and correlation matrix $\Sigma_{Z^*} = D\Sigma_Z D$. The probability of observing a preference for i over j is equal to the probability that the standardized variable is larger than a threshold parameter,

$z_{ij}^* > \tau_{ij}$, where the vector of all threshold parameters is $\tau = -DA\mu$. Maximum likelihood estimation requires the approximation of s integrals of dimension N, the number of preferences expressed by each subject

$$\mathscr{L}(\theta;Y) = \prod_{i=1}^{S} \int_{R_{s12}} \cdots \int_{R_{sn-1n}} \phi_N(\mathrm{z}_s^*;\Sigma_{Z^*})\mathrm{d}\mathrm{z}_s^*,$$

where $\phi_N(\cdot;\Sigma_{Z^*})$ denotes the density of an N-dimensional normal random variable with mean 0 and correlation matrix Σ_{Z^*}, θ denotes the model parameters, *i.e.* the means μ and the elements of the covariance matrix Σ_T, and

$$R_{sij} = \begin{cases} (-\infty, \tau_{ij}) & \text{if } Y_{sij} = 0, \\ (\tau_{ij}, \infty) & \text{if } Y_{sij} = 1. \end{cases}$$

The difficulties in approximating such an integral stimulated the development of alternative estimating techniques.

Maydeu-Olivares [3, 4] proposes a method called limited information estimation. Actually, this name refers to all techniques that base inference on lower dimensional marginals, but in the psychometric literature about paired comparison data the term is generally used to refer to the method proposed by Maydeu-Olivares [3]. The limited information estimation method is performed in three stages. In the first stage the thresholds τ are estimated from the empirical proportions. Let p_{ij} denote the proportions of times in which i is preferred to j, then $\hat{\tau}_{ij} = -\Phi^{-1}(p_{ij})$. In the second stage the entries of the correlation matrix, which are tetrachoric correlations, are estimated employing the sample bivariate proportions of wins. Finally, the model parameters are estimated by minimizing

$$M = \{\tilde{\kappa} - \kappa(\theta)\}^T \hat{W} \{\tilde{\kappa} - \kappa(\theta)\},$$

where $\tilde{\kappa}$ denotes the thresholds and correlation parameters estimated in the first two stages and $\kappa(\theta)$ denotes the corresponding model-based quantities. Finally, $\hat{W}$ is a non-negative definite matrix. Let Ξ denote the asymptotic covariance matrix of $\tilde{\kappa}$, in the literature different proposals have been investigate: $\hat{W} = \hat{\Xi}^{-1}$, $\hat{W} = \{\mathrm{diag}(\Xi)\}^{-1}$ or $\hat{W} = I$, where I is the identity matrix.

3.2 Optimal Estimating Equations

We consider another estimating method which employs only low dimensional marginals. Let $U_{ij} = \sum_{s=1}^{S} Y_{sij}$ denote the number of times object i is preferred to object j and $\pi_{ij} = \mathrm{pr}(Y_{sij} = 1)$, then U_{ij} follows a binomial distribution, $U_{ij} \sim Bin(S, \pi_{ij})$. The estimation of the model assuming independence of the observations corresponds to solving

$$GV^{-1}(U - S\pi) = 0$$

where π is the vector of all probabilities π_{ij}, $i < j = 1, \ldots, n$, G is the matrix of the derivatives of π and V is the covariance matrix of the $U = (U_{12}, \ldots, U_{n-1\,n})$. If independence is assumed, V is a diagonal matrix with entries $\pi_{ij}(1 - \pi_{ij})$.

It is possible to account for correlation in the data through the matrix V. The entries of V are $\mathrm{cov}(U_{ij}, U_{ik}) = E(U_{ij}U_{ik}) - E(U_{ij})E(U_{ik})$. Since preferences of different judges are independent $\mathrm{cov}(U_{ij}, U_{ik}) = S\left\{\mathrm{E}(Y_{sij}Y_{sik}) - \mathrm{E}(Y_{sij})\mathrm{E}(Y_{sik})\right\} = S\left\{\mathrm{pr}(Y_{sij} = 1, Y_{sik} = 1) - \mathrm{pr}(Y_{sij} = 1)\mathrm{pr}(Y_{sik} = 1)\right\}$.

Different approaches are possible. A first alternative consists in estimating the univariate and bivariate probabilities that enter in V by means of the sample univariate and bivariate proportions of wins.

Another possibility is to consider the correlation matrix induced by the latent variable specification. The univariate probabilities can be easily computed as $\mathrm{pr}(Y_{sij} = 1) = \mathrm{pr}(Z^*_{sij} > \tau_{ij}) = \Phi(-\tau_{ij})$, while the bivariate probabilities are computed as $\mathrm{pr}(Y_{sij} = 1, Y_{sik} = 1) = \mathrm{pr}(Z^*_{sij} > \tau_{ij}, Z^*_{sik} > \tau_{ik}) = \Phi_2(-\tau_{ij}, -\tau_{ik}; \rho)$, where $\Phi_2(\cdot, \cdot; \rho)$ denotes the cumulative distribution function of a bivariate normal random variable with mean 0 and correlation ρ. The parameter ρ is the relative element of the correlation matrix Σ_{Z^*}. In this case model parameters can be estimated through a hybrid pairwise likelihood method as the one proposed by Kuk [2].

This method employs only up to bivariate margins as the limited information estimation method commonly employed in the psychometric literature. However, it is appealing because it is based on the optimal linear combination of the unbiased score equations. The method proposed can be seen as a technique for estimating hierarchical models with random effects, but it is computationally easier than other methods and also more robust to the misspecification of the dependence structure. On the other hand, it can be considered an extension of quasi likelihood methods developed for dependent data alternative to generalized estimating equations.

Space constraint prevent the inclusion of results of comparisons between the methods. Simulation results will be presented during the talk along with methodological and practical implications of the two methods and applications to real data.

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