New approach to the identification of the Inverse Weibull model

Giuliana Pallotta and Biagio Palumbo

Abstract This paper proposes a wider application of the Inverse Weibull (IW) distribution. Several generative mechanisms leading to the IW distribution are introduced, offering practitioners a "physical" approach to this model. In order to illustrate the adequacy of the IW model to interpret phenomena from different application areas, some applicative examples are provided and fully discussed.

1 Introduction

The Inverse Weibull (IW) distribution is not widely known and, consequently, scarcely used. On the contrary, important types of mechanism found in Biometry, Reliability and related fields generate random variables having such a distribution. Moreover, failing to identify this distribution may lead to use "similar" models such as Inverse Gaussian (IG) and Lognormal (LN) ones. However, these models, even when appear well fitted to (IW) data, may lead to incorrect assessments concerning, for instance, critical prognoses.

The IW model is referred to with many different names like "Fréchet-type" (Johnson et al. 1995) and "Inverse Weibull" (Erto, 1982, where this name was assigned to it; Johnson et al. 1995).

Its peculiar upside-down bathtub-shaped hazard rate function has been really found in several applications (Erto 1989; Jiang et al. 2003).

Unfortunately, also the IG and the LN models show similarly shaped hazard rate functions. So, besides the best-fitting policy, to allow a preliminary identification of the IW model, we found very useful to take into account the main actual generative mechanisms leading to it. So, this perspective motivates a further insight into the IW model.

1 Giuliana Pallotta, University of Naples Federico II; email: g.pallotta@unina.it
Biagio Palumbo, University of Naples Federico II; email: biagio.palumbo@unina.it

The basic statistical functions of the IW model and their specific properties have been illustrated in Erto (1989). Four generative mechanisms leading to the IW model are reported in Section 2, each one with a related illustrative example. The first three listed mechanisms are introduced and discussed thoroughly in Erto (1989). The last one mechanism is derived from literature. At last, conclusions are given in Section 3.

2 Generative mechanisms with illustrative examples

In order to practically identify the IW distribution as the main candidate to interpret specific phenomena, the knowledge of the generative mechanisms, leading to it, is helpful in a preliminary model selection phase. We found the following principal ones: “Deterioration”, “Stress-Strength”, “Shocks” and “Extreme maximum value” mechanisms. Four real datasets have been considered and are reported in Table 1. For each dataset the adequacy of the IW model has been examined by identifying the specific generative mechanism. In addition, for each dataset, the adequacy is graphically examined in terms of the data scattering around the straight line fitted using the Least Square estimates proposed in Erto (1989). The related correlation coefficients ρ are reported in the last column of Table 1. As expected, also other “similar” models show a significant goodness-of-fit to the above datasets, as shown in Table 2 reporting the Kolmogorov-Smirnov test results. We also derived a novel test based on the Ratio of Maximized Likelihoods (RML) (see Dumonceaux and Antle 1973), starting from that the log-transformation of the IW and the LN distributions into the Type 1 extreme value distribution for maxima and the Normal distribution, respectively. For each dataset the value of the test statistic and the *p*-value are reported in Table 3. As we can see, we could not reject the IW model in favour of the LN one in all the four examples. However, we point out that the power of the RML test, evaluated via Monte Carlo simulation, is not always satisfactory, mostly for small sample sizes.

So the choice of the best model is questionable and the approach via the generative mechanism can be useful to explicit the knowledge of the involved scientific field and it results effective to preliminarily select the IW model as the main candidate to be used for interpretative purposes.

The “Deterioration” mechanism has been illustrated in Erto (1989). This mechanism is found in degenerative phenomena when the deterioration deep reaches a fixed threshold. The related dataset consists of 46 maintenance data on active repair times (in hours) for an airborne communication transceiver. The data were used in Chhikara and Folks (1989). Repair times can be described via the distribution of the (first-passage) time that a Brownian motion with positive drift takes to reach a fixed positive threshold (see, e.g., Sherif and Smith 1980). We can argue that the stress of the repair action (in terms of the maintenance “force” to lead to system repair) is a Weibull random variable. On the other hand, the system resists against the repair “force” via a practically constant strength which is an upper threshold.

The “Stress-Strength” mechanism has been illustrated in Erto (1989). This mechanism can be found in patients with a decreasing vital strength (e.g., because they are subjected to intensive and prolonged chemotherapeutic treatments) and subject to a relapse having a random virulence. The related dataset consists of survival times (in days) of 11 male patients affected by squamous carcinoma in the oropharynx and subjected to radiation therapy alone (Ebrahimi 1993). More specifically, patients show a physical weakening over time (as a consequence of the considerable toxic effects of the radiation treatment). Therefore, they exhibit a decreasing vital strength and the corresponding aggressiveness of the carcinoma results to be random but time independent.

The "Shocks" mechanism has been illustrated in Erto (1989). This mechanism is found in Biometry when the immune system works randomly against antigens or transient defects, and its effectiveness decreases quickly as the incubation time increases (see Le Cam and Neyman 1982, p. 15). The related dataset consists of survival times (in seconds) of 20 insects exposed to a new insecticide (Lee 1992). In this case we can argue that the effectiveness of the immune response decreases very slowly over time. Moreover, the lack of memory of the immune system of insects, discussed in Vilmos and Kurucz (1998), allows modeling their defensive attempts, against the toxic substance, by means of the Poisson distribution. Thus a "shocks" mechanism follows.

The "Extreme maximum value" mechanism is related to the maximum value of a critical non-negative variable (Johnson et al. 1995). The related dataset consists of 25 precipitation data (in inches) from Jug Bridge, Maryland (Folks and Chhikara 1978). The "Extreme maximum value" mechanism can provide a reasonable comprehension of the phenomenon. This conforms to a general statistical modelling of precipitation data that appeared recently in Vovoras and Tsokos (2009). They argue that, among the three extreme value distributions, the Fréchet is the most adequate for rainfall data since it is a heavy tailed distribution.

3 Concluding remarks

In this paper some generative mechanisms leading to the IW distribution are introduced. These mechanisms, on the basis of the knowledge of the "physics", enable one to preliminarily choose the IW model as the main candidate to interpret the involved phenomenon, such as a disease, rather than exclusively on the basis of a routine goodness-of-fit analysis of field data. The proposed strategy makes more reliable the IW model identification compared to other "similar" models such as the IG and the LN ones.

Table 1: Datasets used in the illustrative examples and the correlation coefficients ρ

	Dataset	**Size**	**Data**	ρ
(a)	Repair times	$n = 46$	0.2 0.3 0.5 0.5 0.5 0.5 0.6 0.6 0.7 0.7 0.7 0.8 0.8 1.0 1.0 1.0 1.0 1.1 1.3 1.5 1.5 1.5 1.5 2.0 2.0 2.2 2.5 2.7 3.0 3.3 3.3 3.7 4.0 4.0 4.5 4.7 5.0 5.4 5.4 7.0 7.5 8.8 9.0 10.3 22.0 24.5	0.98
(b)	Carcinoma	$n = 11$	167 238 296 324 351 372 374 404 541 560 943	0.98
(c)	Insects	$n = 20$	3 5 6 7 8 9 10 10 12 15 15 18 19 20 22 25 28 30 40 60	0.97
(d)	Precipitation	$n = 25$	1.01 1.11 1.13 1.15 1.16 1.17 1.17 1.20 1.52 1.54 1.54 1.57 1.64 1.73 1.79 2.09 2.09 2.57 2.75 2.93 3.19 3.54 3.57 5.11 5.62	0.96

Table 2. Kolmogorov-Smirnov test statistic D_n and corresponding *p*-value for each dataset using the Inverse Weibull, the Inverse Gaussian and the Lognormal models.

			Inverse Weibull		**Inverse Gaussian**		**Lognormal**	
	Dataset	**Size**	D_n	***p*-value**	D_n	***p*-value**	D_n	***p*-value**
(a)	Repair Times	$n = 46$	0.08	0.92	0.07	0.98	0.09	0.80
(b)	Carcinoma	$n = 11$	0.16	0.93	0.17	0.93	0.16	0.93
(c)	Insects	$n = 20$	0.13	0.90	0.10	0.99	0.09	0.99
(d)	Precipitation	$n = 25$	0.16	0.93	0.15	0.67	0.14	0.66

Table 3 Test statistic $(RML)^{1/n}$ and the *p*-value with $H_0 : IW$ vs $H_1 : LN$

	Dataset	**Size**	$(RML)^{1/n}$	***p*-value**
(a)	Repair Times	$n = 46$	0.639	0.999
(b)	Carcinoma	$n = 11$	0.828	0.889
(c)	Insects	$n = 20$	0.817	0.921
(d)	Precipitation	$n = 25$	0.804	0.947

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