PDE penalization for spatial fields smoothing

Laura Azzimonti, Laura M. Sangalli, Piercesare Secchi, Silvia Romagnoli and Maurizio Domanin

Abstract We propose a novel functional data analysis technique for surface estimation, based on a bivariate generalization of smoothing splines. This technique is developed in order to estimate the blood velocity field in a section of a carotid artery, using data provided by eco-color doppler. The proposed method is especially well suited for those applications where there is a prior knowledge of the phenomenon under study coming from the physics or the physiology. The method can deal both with pointwise data and areal data. Moreover, the method can estimate surfaces over irregularly shaped regions with general conditions at the boundary of the domain.

Key words: functional data analysis, spatial smoothing, areal data, finite elements

1 Introduction and motivating problem

In this work we propose a novel smoothing technique for surface estimation, based on a bivariate generalization of smoothing splines. The proposed technique is applied to the estimation of the blood-flow velocity field in a section of a carotid artery, using data provided by eco-color doppler. This study is carried out within the project *MACAREN@MOX* (*MAthematichs for CARotid ENdarterectomy @ MOX*), which aims at developing numerical and statistical tools for the study of the atherosclerotic plaque in human carotids. The estimation of the blood velocity field over the whole carotid section is of interest since the blood-flow regime could help in explaining the presence and histological properties of atherosclerotic plaques; see for instance

Laura Azzimonti, Laura M. Sangalli and Piercesare Secchi
MOX - Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy e-mail: laura.azzimonti@mail.polimi.it

Silvia Romagnoli and Maurizio Domanin
U.O. di Chirurgia Vascolare Fondazione I.R.C.C.S. Ca' Granda Ospedale Maggiore Policlinico, Milano and Dipartimento di Scienze Chirurgiche Specialistiche, Università di Milano

[3].
We estimate the blood flow velocity field starting from the eco doppler signals measured on the section of the carotid artery located 2 cm before the carotid bifurcation. Figure 1, left panel, displays one such signal. The eco doppler measures the velocity of the blood particles within a small volume, named beam (see [9]). In this clinical study, the beams are located in a cross shape pattern on the carotid section, as shown in the central panel of Figure 1. The eco doppler signals are measured over time and, for each time, the signal represents the histogram of the velocity of the blood cell in the beam. In this work we fix the time at the systolic peak and, for each beam, we summarize the corresponding histogram of the velocity at this time by its first moment (i.e. we only consider the mean velocity in each beam). The central panel of the Figure 1 displays the mean velocity of blood particles over each beam, at the systolic peak. In the systolic phase the real geometry of the carotid section can be well approximated by a circle but it should be noticed that the proposed method can deal with more complex domains, allowing the use of patient-specific geometries, for example during the diastolic phase when the real geometry is not circular. It is also important that the method can comply with specific conditions at the boundary of the domain, i.e. at the arterial wall. In fact, in order to obtain physiological meaningful results, it is necessary to require the blood-flow velocity at the arterial wall to be zero due to the friction of the blood cells with the wall.
Since classical methods for surface estimation, as for instance thin-plate splines, tensor product splines and kriging, are not able to take into account the shape of the domain and cannot comply with specific conditions at the domain boundary, we extended the method described in [7], [6] and [8] that generalizes smoothing splines to bidimensional domains. The novel technique here proposed, described in more details in [1] and [2], is particularly well suited for applications where there is a physical or physiological prior knowledge of the phenomenon under study.

2 Model

Consider a bounded and regular domain $\Omega \in R^2$ and observations z_i, for $i = 1, \ldots, n$, that represent the mean of a surface $s : \Omega \rightarrow R$ on the subdomains $D_i \subset \Omega$. The model that we consider is the following

$$z_i = \frac{1}{|D_i|} \int_{D_i} s(\mathbf{x}) d\mathbf{x} + \varepsilon_i \qquad (1)$$

where ε_i are independent errors with zero mean and constant variance, s is the surface that we want to estimate and $|D_i|$ is the Lebesgue measure of the subdomain D_i. In our application we have a physical and physiological knowledge of the problem, in particular on the shape of the surfaces and the conditions on the boundary, and we want to take advantage of this information in the surface estimation. In order to estimate s we propose to minimize the penalized sum-of-square-error functional

$$J(s) = \sum_{i=1}^{n} \left| \int_{D_i} (s(\mathbf{x}) - z_i)\, d\mathbf{x} \right|^2 + \lambda \int_{\Omega} (Ls - g)^2 \qquad (2)$$

requiring that the surface s satisfies the imposed boundary conditions. The first term is a least-square-error functional for areal data over subdomains D_i while the second one is a roughness penalty term. The roughness term of the functional penalizes the misfit of a partial differential operator known to model to some extent the phenomenon under study. The partial differential equation (PDE) in the penalizing term is described by means of a second order elliptic operator L and a forcing term $g \in L^2(\Omega)$. The operator L is a linear operator that can include first and second order differential operators, for example

$$Ls = -div(K\nabla s) + \mathbf{b} \cdot \nabla s + cs \qquad (3)$$

with parameters that can be spatially varying on Ω. Setting $K = I$, $\mathbf{b} = \mathbf{0}$, $c = 0$ and $g = 0$, and considering pointwise observations (i.e., $I_{D_i} = \delta_{\mathbf{x}_i}$), we obtain the special case, described in [7], in which the Laplacian Δs, which is a measure of local curvature of the surface s, is penalized.

Theorem 1. *The solution of the minimization of the functional $J(s)$ defined by equation (2) over the Sobolev space of surfaces $H^2(\Omega)$ with proper boundary conditions, exists and is unique*

The proof is detailed in [2]. Since the solution of the system is not analytically computable, we approximate the solution, similarly to [7], by means of the Finite Element method, see [5]. Thanks to the Finite Element approximation we obtain a linear estimator for the surface and we can derive classic inferential results as pointwise confidence bands and predictions intervals, see [2] and [8] for more details. The proposed method is currently implemented in FreeFem++ [4] and `R`.

3 Blood-flow velocity field

In order to estimate the velocity field of the blood in the carotid section we minimize the functional $J(s)$ defined in (2), with L defined as in (3) where the diffusion tensor K smooths the surface along circles and the transport parameter $\mathbf{b}$ smooths the observations in the radial direction. The right panel of Figure 1 displays the obtained velocity field estimate; such a velocity profile is justified by the curvature of the carotid artery. The estimated velocity field obtained penalizing the misfit of an elliptic PDE is much more physiological and less influenced by the location of the data than the one obtained penalizing only a measure of the local curvature, as shown in [1]. The proposed spatial smoothing method is also more flexible than classic parametric models, in fact, it can deal with general velocity fields and it can capture for example asymmetries and reversion of the fluxes. The estimation of the velocity field over the whole section provides richer information for diagnostic purposes than the original observations; these information will be used for a correct

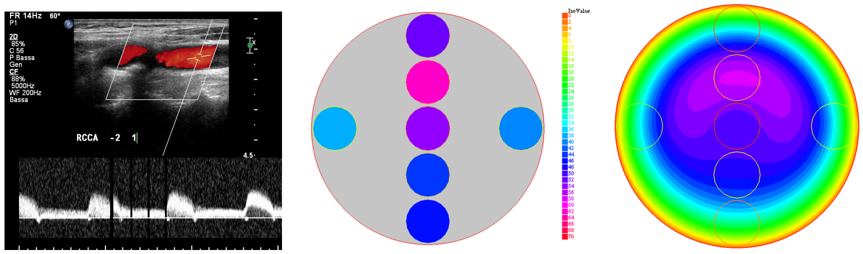

Fig. 1 Left panel: eco-color doppler image corresponding to the central point of the carotid section located 2 cm before the carotid bifurcation. Central panel: mean velocity on the beams during the systolic phase. Right panel: smoothing estimate of the velocity field.

understanding of the interactions between blood fluid-dynamics and presence and properties of atherosclerotic plaques.

We are extending the proposed method to include the time dimension, in order to model time-dependent surfaces. Such extension would allow to study how the blood-flow velocity field varies over the time of the heart beat.

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References

1. Azzimonti, L., Domanin, M., Sangalli, L.M., Secchi, P.: Surface estimation via spatial spline models with PDE penalization. In: Proceeding of S.Co. 2011 Conference (2011).
2. Azzimonti, L., Domanin, M., Sangalli, L.M., Secchi, P.: PDE penalization for spatial fields smoothing. In preparation (2012).
3. Moyle, K. R., Antiga, L., Steinman, D. A.: Inlet conditions for image-based CFD models of the carotid bifurcation: is it reasonable to assume fully developed flow? J Biomech Eng. **128**, 371–9 (2006).
4. Pironneau, O., Hecht, F., Le Hyaric, A., Morice, J.: FreeFem++ Software version 3.13 (2011) Available at www.freefem.org
5. Quarteroni, A., Sacco, R., Saleri, F.: Numerical mathematics, second edition. Springer (2007).
6. Ramsay, J. O., Ramsay, T., Sangalli, L. M.: Spatial Functional Data Analysis. Recent Advances in Functional Data Analysis and Related Topics, Contributions to Statistics, Physica-Verlag Springer, 269–276 (2011).
7. Ramsay, T.: Spline Smoothing over Difficult Regions. J. R. Stat. Soc. Ser. B Stat. Methodol., **64**, 307–319 (2002).
8. Sangalli, L. M., Ramsay, J. O., Ramsay, T.: Spatial spline regression models. Tech. Rep. 08/2012, MOX – Dipartimento di Matematica, Politecnico di Milano (2012). http://mox.polimi.it/it/progetti/pubblicazioni/quaderni/08-2012.pdf
9. Schaberle, W.: Ultrasonography in vascular diagnosis. Springer (2005).