Spatial smoothing over non-planar domains

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Abstract We describe a novel Functional Data Analysis method for smoothing spatially distributed data. We address the case where spatial data occur on a non-planar bi-dimensional domain. In particular, we are interested in surface domains embedded in a 3D space. Our approach, conformally maps the original 3D domain to a region of the plane. Then existing spatial smoothing techniques are suitably modified to account for the domain deformation described by the conformal map. The application driving the proposed approach is the smoothing of hemodynamic data, such as wall shear stress or blood pressure, on the wall of a carotid artery.

Key words: Functional data analysis, penalized smoothing, finite elements.

1 Introduction

In this work, we are interested in smoothing spatial data that occur over non-planar domains. Unfortunately, only few methods are available to deal with this type of data structure. One such model is presented in [1] where eigenfunctions of the Laplace-Beltrami operator of the surface domain are used to construct a heat kernel smoothing method. Here, we adopt a Functional Data Analysis approach, and propose a regression method that efficiently handles data over non-planar domains. The key idea is to flatten the original surface domain by means of a conformal map; the important property of a conformal map is that it preserves the angles and shapes of the original surface domain in the planar domain. On the planar domain, it is possible to use the method for spatial smoothing proposed in [6] and [7], suitably modifying this method to account for the deformation of the domain.

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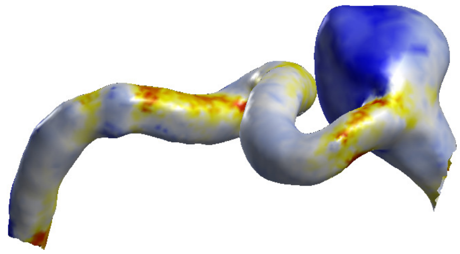

Fig. 1 Wall shear stress modulus at the systolic peak on real inner carotid artery geometry affected by aneurysm (data of the AneuRisk project, http://ecm2.mathcs.emory.edu/aneurisk).

2 Data and model

Consider a set of n data locations $\{\mathbf{x}_i = (x_i, y_i, z_i);\, i = 1\ldots, n\}$ that lie on a regular surface Σ embedded in a $\mathbb{R}^3$. Let w_i be the value of a real valued random variable of interest observed at the point $\mathbf{x}_i$. The model we want to consider is

$$w_i = f(\mathbf{x}_i) + \varepsilon_i \qquad\qquad i = 1, \ldots, n \tag{1}$$

where ε_i, $i = 1, \ldots, n$ are observational errors and f is the function we aim to estimate. For example, consider the shear stress generated by blood-flow on the wall of an internal carotid artery affected by an aneurysm, as illustrated in Figure 1 (for a detailed description of the data and the applied problem see [5]; the data are part of the AneuRisk project). In this example, the variable of interest is the wall shear stress observed over the non-planar wall of the carotid artery. These types of data structures occur in a number of different applications, not only in the medical field but, e.g., also in environmental and geostatistical studies.

In the standard case of functions over planar bi-dimensional domains [6] and [7] propose to estimate the function f by minimizing a penalized sum of squared error functional. In particular, the penalty term is proportional to the integral, over a planar domain of interest, of the squared Laplacian of the function. The Laplacian locally measures the curvature of the function and hence this choice for the penalty yields an estimate of f that approximates the observed data without being bumpy. By analogy, in [3] we propose to estimate f in (1) by minimizing the following penalized sum of squared error functional

$$J_\lambda(f(\mathbf{x})) = \sum_{i=1}^{n} (w_i - f(\mathbf{x}_i))^2 + \lambda \int_\Sigma (\Delta_\Sigma f(\mathbf{x}))^2 \; d\mathbf{x}, \tag{2}$$

where Δ_Σ is the Laplace-Beltrami operator for functions defined over the surface Σ. The Laplace-Beltrami operator is indeed the generalization of the common Lapla-

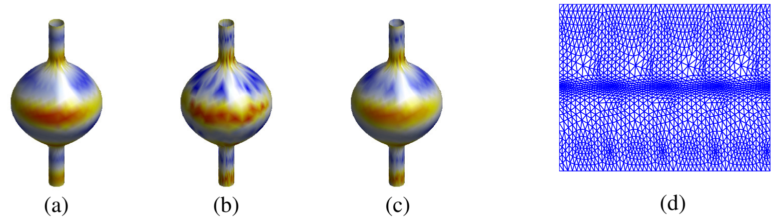

Fig. 2 (a) Test function; the surface domain is approximated by a 3D triangular mesh. (b) The test function with simulated noise added at each data point. (c) Estimate obtained with the proposed method. (d) The planar triangulated domain, conformally equivalent to the surface domain shown in left panel.

cian: it can be used to operate on functions defined on surfaces in Euclidean spaces (see, e.g., [2]).

To minimize the functional in (2) by exploiting the planar techniques in [6] and [7], we propose reducing (2) to a problem over a planar domain. To do this, we flatten Σ by means of a conformal map X. In particular, we define X as the map

$$\begin{aligned} X : \Omega &\to \Sigma \\ \mathbf{u} = (u,v) &\mapsto \mathbf{x} = (x,y,z) \end{aligned} \tag{3}$$

where Ω is an open, convex and bounded set in $\mathbb{R}^2$. Denote by $X_u(\mathbf{u})$ and $X_v(\mathbf{u})$ the column vectors of first order partial derivatives of X with respect to u and v and define the metric tensor as the following symmetric positive definite matrix

$$G(\mathbf{u}) := \begin{pmatrix} \|X_u\|^2 & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \|X_v\|^2 \end{pmatrix}(\mathbf{u}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(\mathbf{u})$$

where $g_{12} = g_{21}$, $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product of two vectors and the corresponding norm is denoted $\|\cdot\|$. Set $\mathscr{W}(\mathbf{u}) := \sqrt{\det(G(\mathbf{u}))}$, and denote by $G^{-1}(\mathbf{u}) = \{g^{ij}(\mathbf{u})\}_{i,j=1,2}$ the inverse of $G(\mathbf{u})$. Using this notation, for a function $f \circ X \in \mathscr{C}^2(\Omega)$, the Laplace-Beltrami operator of the surface Σ can be expressed as

$$\Delta_\Sigma f(\mathbf{x}) = \frac{1}{\mathscr{W}(\mathbf{u})} \sum_{i,j=1}^{2} \partial_i (g^{ij}(\mathbf{u})(\mathscr{W}(\mathbf{u})\partial_j f(X(\mathbf{u}))).$$

where $\mathbf{u} = X^{-1}(\mathbf{x})$. In [3], we show that (2) can be equivalently expressed as the following problem over the planar domain Ω:

$$\begin{aligned} &J_\lambda(f(X(\mathbf{u}))) \\ &= \sum_{i=1}^{n} (w_i - f(X(\mathbf{u}_i)))^2 + \lambda \int_\Omega \left[\frac{1}{\mathscr{W}(\mathbf{u})} \sum_{i,j=1}^{2} \partial_i (g^{ij}(\mathbf{u})\mathscr{W}(\mathbf{u})\partial_j f(X(\mathbf{u})) \right]^2 \mathscr{W}(\mathbf{u}) d\mathbf{u} \end{aligned} \tag{4}$$

where $X(\mathbf{u}_i) = \mathbf{x}_i$. This problem turns out to be a modification of the estimation problem solved in [6] and [7].

From a computational view point, both the calculation of the conformal flattening map in (3) and the solution to problem (4) are carried out resorting to a Finite Element approach, in particular non-planar and planar finite elements respectively. Finite elements provide local basis for piecewise polynomial surfaces over a triangulation of the domain. To compute (3) we use the method described in [4]. Figure 2 shows the flattening of a test surface domain using non-planar finite elements; panel (a) shows the starting non-planar domain approximated by a fine 3D triangular mesh, and panel (d) displays the conformally equivalent planar triangulated domain. The problem is thus ready for implementation via the estimation method in [6] and [7] based on planar finite elements, that has been suitably modified to account for the deformation of the domain by the conformal map. Also illustrated in Figure 2 is a simulation on the non-planar domain. The function $f(x,y,z) = \sin(2\pi z) + xy + 1$ where x,y,z are restricted to non-planar domain is used as a test function (shown as a colormap on the non-planar domain in panel (a)). Noisy data are generated by evaluating the test function at each vertex of 3D mesh and then adding independent normal errors with mean zero and standard deviation 0.25. Panel (b) shows the noisy data and panel (c) shows the estimate obtained with the proposed method, setting $\lambda = 0.0014$. The proposed method does a good job of estimating the test function thus demonstrating the future promise of the proposed method for smoothing spatial data over non-planar domains.

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References

1. Chung, M.K., Robbins, S.M., Dalton, K.M., Davidson, R.J., Alexander, A.L., Evans, A.C. (2005), "Cortical thickness analysis in autism with heat kernel smoothing," *NeuroImage,* 25, 1256-1265.
2. Dierkes, U., Hildebrant, S., K*ü*ster, A., and Wohlrab, O. (1992), "Minimal Surfaces (I)", *Springer*.
3. Ettinger, B.D., Perotto, S., Sangalli, L.M. (2012), "Spatial smoothing over non-planar domains," in preparation.
4. Haker, S., Angenent, S., Tannenbaum, A., Kikinis, R. (2000), " Nondistorting flattening maps and the 3-D visualization of colon CT images", *IEEE Trans. Med. Imag.,* 19, 665-670.
5. Passerini, T., Sangalli, L.M., Vantini, S., Piccinelli, M., Bacigaluppi, S., Antiga, L., Boccardi, E., Secchi, P., and Veneziani, V. (2012), "An Integrated Statistical Investigation of Internal Carotid Arteries of Patients affected by Cerebral Aneurysms," *Cardiovascular Engineering and Technology*, doi: 10.1007/s13239-011-0079-x.
6. Ramsay, T. (2002), "Spline Smoothing over Difficult Regions," *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 64, 307-319.
7. Sangalli, L.M., Ramsay, J.O., and Ramsay, T.O. (2012), "Spatial Spline Regression Models," Tech. rep. N. 08/2012, MOX, Dipartimento di Matematica F.Brioschi, Politecnico di Milano, available at http://mox.polimi.it/it/progetti/pubblicazioni. Submitted